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STATISTICAL APPROXIMATIONS AND THE PHYSICS OF TURBULENCE

Jackson R. Herring
N.C.A.R.
Boulder
Colorado 80307
Email: herring@ucar.edu

ABSTRACT

For nearly half a century, statistical closures have been used to study turbulent flows in a variety of circumstances. Such methods are formulated in terms of average quantities, and should be simpler to compute than Navier–Stokes. However, comparisons of their predictions to experiments and to numerical simulations suggest that the accuracy of such theories are limited by the existence in the flows of isolated structures, whose characteristics depend on the nature of the flow (whether buoyantly active, neutral, or chemically reactive) and on flow boundaries. Nonetheless, their application to a variety of problems suggests that they can give valuable insights into flow characteristics. In this paper we review several applications of closure to practical problems, and note their successes and failures. The sections to follow discuss the following topics: a short derivation of the theory, stressing its physical assumptions; the use of the theory in yielding insights into the spectral form of large eddy simulations; the evolution of passive scalar spectra and their dependence on initial conditions and Prandtl numbers; and a brief discussion of the application of the theory to stably stratified turbulence. Finally, we conclude with a discussion of the impact of statistical theory on turbulence research and the prospects for improvements capable of dealing with structural effects.

INTRODUCTION

In a statistical theory the goal is to predict $U_{ij}(\mathbf{x},\mathbf{x}',t,t') \equiv \langle \mathbf{u_i}(\mathbf{x},t)\mathbf{u_j}(\mathbf{x}',t')\rangle$, where $\langle\cdot\rangle$ denotes an ensemble average. (More generally, we may wish to predict $\rho(\mathbf{u_1},\mathbf{u_2},\mathbf{u_3},\cdots)$, where ρ denotes the probability density that $\mathbf{u}$ will be $\mathbf{u}_1$ at $\mathbf{r}_1$, etc.). We shall focus exclusively on the simpler goal, whose solution will enable us to compute energy spectra and fluxes. Consider then the incompressible Navier Stokes equations, which, to facilitate a derivation of equations for $U_{ij}(\mathbf{x},\mathbf{x}',t,t')$, we write down twice: once at $\mathbf{r},t$ and once at $\mathbf{r}',t'$:

$$(\partial_t - \nu\nabla^2)\mathbf{u} \equiv L\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p \tag{1}$$

$$(\partial_{t'} - \nu\nabla'^2)\mathbf{u}' \equiv L'\mathbf{u}' = -\mathbf{u}'\cdot\nabla'\mathbf{u}' - \nabla p' \tag{2}$$

Here, the pressure, p, is a bi–linear functional of $\mathbf{u}$ that assures incompressibility:

$$-\nabla^2 p = \nabla\cdot\{\mathbf{u}\cdot\nabla\mathbf{u}\} \tag{3}$$

The following is a symbolic sketch of a derivation of the closure. What emerges is a modelization of the fate of $\partial_t\mathbf{u}$ as comprised of two parts: a random force derivable from the pressure gradient and advection terms, and a generalized eddy viscosity. As will hopefully be apparent from the presentation, no empirical constants are indicated, and the procedure is applicable to any incompressible flow, inhomogeneous and anisotropic. The equations of motion for $U_{ij}(\mathbf{x},\mathbf{x}';t,t')$ are the Direct Interaction Approximation of Kraichnan, whose proper derivation are to be found in (Kraichnan 1959, 1964a).

The equation evolving U_{ij} may be recast by multiplying (1) by (2):

$$LL'\langle\mathbf{u}(\mathbf{x},t)\mathbf{u}(\mathbf{x}',t')\rangle = \langle(NL)\times(NL')\rangle \tag{4}$$

Here, we denote $NL \equiv -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p$, so that the right–hand side of (4) is fourth order in $\mathbf{u}$. If the statistics of $\mathbf{u}(\mathbf{x},t)$ are Gaussian, this fourth order moment may be reduced to products of second order covariances through the use of the Gaussian relationship,

$$\langle \mathbf{u}\,\mathbf{u}'\,\mathbf{u}''\,\mathbf{u}'''\rangle = \langle \mathbf{u}\,\mathbf{u}'\rangle\langle \mathbf{u}''\,\mathbf{u}'''\rangle + \text{et cyc.} \tag{5}$$

If (5) is used in (4) we have a closed system for $U_{ij}(\mathbf{x},\mathbf{x}',t,t')$. The problem with this Gaussian assumption is that it cannot conserve energy $E(\mathbf{x},t) = (1/2)\sum_i U_{ii}(\mathbf{x},\mathbf{x},t,t)$ even for homogeneous flows (E independent of $\mathbf{x}$) and for $\nu \to 0$. Qualitatively, this fault results from modeling NL as if it were a Gaussian forcing of $\mathbf{u}$; such a forcing must result in an increase of the system's energy. The statistical theory we now describe modifies $L \equiv (\partial_t - \nu\nabla^2) \to \hat{L}$ in such a way that energy conservation is restored. To see how this may be done, we rewrite (4) in a form more convenient to examine energy conservation:

$$(\partial_t - \nu\nabla^2)\langle \mathbf{u}(\mathbf{x},t)\mathbf{u}(\mathbf{x}',t')\rangle = \{L'\}^{-1}\langle (NL)\times(NL')\rangle \tag{6}$$

Here, $L^{-1}f(\mathbf{x},t)$ means

$$\int_0^t dt' \int d\mathbf{x}' G(\mathbf{x},\mathbf{x}';t,t')f(\mathbf{x}',t') \tag{7}$$

where

$$(\partial_t - \nu\nabla^2)G(\mathbf{x},\mathbf{x}';t,t') = \delta(\mathbf{x},\mathbf{x}')\delta(t,t') \tag{8}$$

We use a notation such that $(\partial_t - \nu\nabla^2)^{-1} \equiv G$, so that (8) has the alternate matrix representation:

$$LG = I \tag{9}$$

To determine the form of $\hat{L} \equiv L + \delta L$ needed to restore energy conservation, we change (6) to;

$$(\partial_t - \nu\nabla^2 + \delta L)\langle \mathbf{u}\,\mathbf{u}'\rangle = \{\hat{L}'\}^{-1}\langle (NL)\times(NL)'\rangle_G \tag{10}$$

Here, the notation $\langle\cdot\rangle_G$ means to evaluate the moments of $\mathbf{u}$ encased in $\langle\rangle$ as if Gaussian (*i.e.:* as prescribed by (x5)). If (10) has the property that $\dot{E} \equiv \partial_t \int d\mathbf{x}\langle \mathbf{u}^2\rangle(\mathbf{x},t) = 0$ (for $\nu = 0$) it is clear that

$$\delta L \sim \{\hat{L}\}^{-1}Q\langle \mathbf{u}\otimes\mathbf{u}\rangle \tag{11}$$

in order for the right–hand side contribution of (10) to $\dot{E}$ to cancel with the appropriate contraction of $\delta L\langle \mathbf{u}\,\mathbf{u}'\rangle$. Here, Q is a matrix that renders $\delta L\langle \mathbf{u}\otimes\mathbf{u}\rangle$ to have the same dimensions as NL, and $\mathbf{u}\otimes\mathbf{u}$ is the matrix indexed by vector, spatial, and temporal dependence of $\mathbf{u}$: $u_i(\mathbf{x},t)u_j(\mathbf{x}',t')$. We shall not give details of how the $\sim$ relation in (11) is replaced by an $=$, except to note that for a conservative non–linear system (such as Navier–Stokes) this may be done without ambiguity (no arbitrary coefficients). The point is that conservation must be obtained for arbitrary realizable $U_{ij}(\mathbf{x},\mathbf{x}';t,t')$, and this is sufficient to determine $\delta\hat{L}$. Finally, we may extract an equation for $\hat{L} \equiv \hat{G}$ which replaces the viscous Green's function defined by (8):

$$\{L + GQ\langle \mathbf{u}\otimes\mathbf{u}\rangle\}G = I \tag{12}$$

To summarize, we started with the assumption that the nonliner terms, (NL) in Navier–Stokes can be thought of as a random (Gaussian) forcing of $\mathbf{u}$. But if this has any validity, energy conservation forces an introduction of a generalized eddy viscosity, δL. The modeling comes under the generalized category of Langevin theory, with δL a dynamical resistance. The right–hand side of (10) is then the average of the square of the random force times the length of time this force acts $(\hat{L})^{-1}$.

The procedure is general in the sense that it can include flows with boundaries, and flows with mean flow components, such as shear flow and convection. In these cases, rapid distortion terms (interactions of mean fields with fluctuations of zero mean) are included without approximation. The theory is guaranteed realizable, at least in the sense of preserving positivity of energy spectra and relevant Schwartz inequalities for covariances. There remains the question of in what sense such methods aid in understanding and predicting real flows.

Before discussing any results, we should remark that (10) and (11) evolve Eulerian covariances, and as such they are known to carry a fundamental problem in that a spurious large scale sweeping effect is contained in (11), whose effect is to modify the energy spectrum, $E(k,t)$ associated with $\langle u_i(\mathbf{x},t)u_i(\mathbf{x}',t)$,

$$E(k) = \frac{1}{2}\int d(\mathbf{x}-\mathbf{x}')U_{ii}(\mathbf{x}-\mathbf{x}',t,t)\exp(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')) \tag{13}$$

from $\sim k^{-5/3}$ to $\sim k^{-3/2}$. The problem may be avoided by formulating the problem in Lagrangian variables, and this has been done successfully in a number of related ways (Kraichnan 1964b, 1971; Kaneda 1981; and Kida and Gotoh 1997). We shall not discuss these more satisfactory methods in any detail, except to note that the main forcing for the Lagrangian methods is simply ∇p. The basic idea of modeling the perturbation theory on random force–eddy viscosity remain in these more sophisticated formulations.

From the practical point of view, one may ask, why trade in Navier–Stokes for a statistical theory (10)–(12) which seems much more complicated than the former? A part of the answer is that whereas $\mathbf{u}(\mathbf{x},t)$ fluctuates rapidly in its arguments, $U_{ij}(\mathbf{x},\mathbf{x}',t,t')$ generally is a much smoother function of its arguments because of the averaging done to obtain it. In addition, for homogeneous problems, U is a function only of $\mathbf{x}-\mathbf{x}'$, and for stationary flows, a function of $t-t'$.

Another question of a practical nature is why not rely on single point modeling such as that described in the monograph by Schiestêl (1998), (as championed by Launder and colleagues)? Single point closures clearly simpler, with direct physical interpretations of the various terms modeled. Such methods are most useful for problems for which a single length scale suffices: but for flows in which different scales have differing underlying physics, they are valid only to the extent that they incorporate these length scales, and this requires an extension beyond what is usually posited. Two-point closures provide such an extension; they actually provide a continuum of scales in the Fourier sense of (13). Moreover, they contain no arbitrary constants, and are guaranteed realizable. Finally, we note that single point closures may be derived–with suitable approximations–from two point closures. Considerable progress on this front has been made by Yoshizawa (1981, 1998). As noted above, such a derivation includes new terms for flows in which multiple length scales are needed.

Before leaving this section, we show how the statistical theory may be used to indicate inertial ranges, using the simplified form that emerges for isotropic turbulence. We have indicated that the rate of change of $U_{ij}(k,t)$ is basically derived from generalized Brownian motion mechanics, in which $\dot{U}_{ij}$ is proportional to the average square of the force acting on U_{ij} times the length of time this force acts. For the energy spectrum, $E(k,t)\sim 2\pi k^2 U_{ii}(k,t)$ this features translates roughly to:

$$\dot{E}(k,t)+2\nu k^2E(k,t)=T(k,t)\sim\int_0^k p^2dpE(p)\theta(k)E(k)\{\cdots\} \tag{14}$$

The factor $\{\cdots\}$ is non dimensional and assures $\int_0^\infty T(k,t)=0$. The time–scale $\theta(k)$ for isotropic turbulence is–perforce–comprised of $E(k)$, and k alone. Equation (14) states that the squared strain field $\int_0^k p^2dpE(p)$ is the square of the (random) force that lasts (on average time $\theta=1/\sqrt{k^3E(k)}$). Equating $\int_0^k dk'\dot{E}(k')$ to the energy dissipation ε gives

$$\varepsilon\sim k^3E^2(k)\frac{1}{\sqrt{k^3E(k)}},\ E(k)\sim\varepsilon^{2/3}k^{-5/3} \tag{15}$$

We may use (15) for the energy spectrum to obtain an equation for the decay of total energy, $E_{tot}=\int_0^\infty dkE(k,t)$ (Kolmogorov 1942). The analysis assumes $E(k,t)=Ak^4$, $k\leq k_0(t)$, and $E(k,t)\sim\varepsilon^{2/3}k^{-5/3}$, $k\geq k_0(t)$, with continuity at $k_0(t)$. According to (15) we have $E_{tot}(t)=C\varepsilon^{2/3}k_0^{2/3}(t)$, and by definition, $dE_{tot}/dt=-\varepsilon$, which implies

$$E_{tot}(t)\sim t^{-10/7} \tag{16}$$

provided A is constant, which is consistent with the permanence of large eddies, as advocated by Batchelor (1957). A more detailed examination of this point by Lesieur and Schertzer (1978) using a version of the statistical theory of the form (19) shows A to be a weak function of time so that the power $10/7$ should be modified to 1.37.

STATISTICAL THEORY AND LARGE EDDY SIMULATIONS.

LES may be formulated in a variety of ways, but one simple and precise manner is in terms of a spectral closure, as in Kraichnan (1976) and Chollet and Lesieur (1981). In this case, it is convenient to formulate both statistical theory and LES in the wave number domain. In what follows, we treat homogeneous, isotropic flows for simplicity. We decompose $\mathbf{u}(\mathbf{x},t)$

$$\mathbf{u}(\mathbf{x},t)=\sum_{|\mathbf{k}|<k_{max}}\exp(\mathbf{k}\cdot\mathbf{x})\mathbf{u}(\mathbf{k},t) \tag{17}$$

Here, we regard $\mathbf{k}$ as a discrete set, and denote k_{max} as larger than the Kolmogorov wave number, so that $\mathbf{u}$ is well resolved. We then ask if there is an eddy viscosity, that if applied to Fourier modes $\mathbf{u}(\mathbf{k},t)$ on a reduced domain $0\leq k\leq k_c$ (called hereafter D), yields velocity amplitudes whose spectra on D are identical to a spectrum that would be found for an unreduced or full domain, ($0\leq k\leq k_{max}$). We call the reduced domain velocity amplitude $\mathbf{v}(\mathbf{k},t)$, and the full domain velocity field $\mathbf{u}(\mathbf{k},t)$. In such a formulation, all that is claimed about $\mathbf{v}(\mathbf{k},t)$ is that its covariance is related to that of the untruncated $\mathbf{u}(\mathbf{k},t)$ in the manner just outlined: $\langle\mathbf{u}(\mathbf{k},t)\mathbf{u}(-\mathbf{k},t)\rangle=\langle\mathbf{v}(\mathbf{k},t)\mathbf{v}(-\mathbf{k},t)\rangle$, $|\,\mathbf{k}\,|\leq k_c$, $\mathbf{v}(\mathbf{k},t)=0, |\,k\,|\geq k_c$. Thus, no claim is made that the amplitude, $\mathbf{v}(\mathbf{k},t)$ contains details of the actual flow. In particular, the reduced system's skewness (which measures the net energy transfer out of the energy containing range into the dissipating range) will be much smaller for the reduced system.

The LES program just outlined assumes we know the solution to Navier–Stokes on the full domain, but for large Reynolds number flows such knowledge is as yet far beyond analytic or computational capacity. But we can apply the program cleanly to a statistical theory, which is designed to capture the variance–level information of Navier–Stokes. It is the application of the program to this surrogate system that constitutes the subject of the present section.

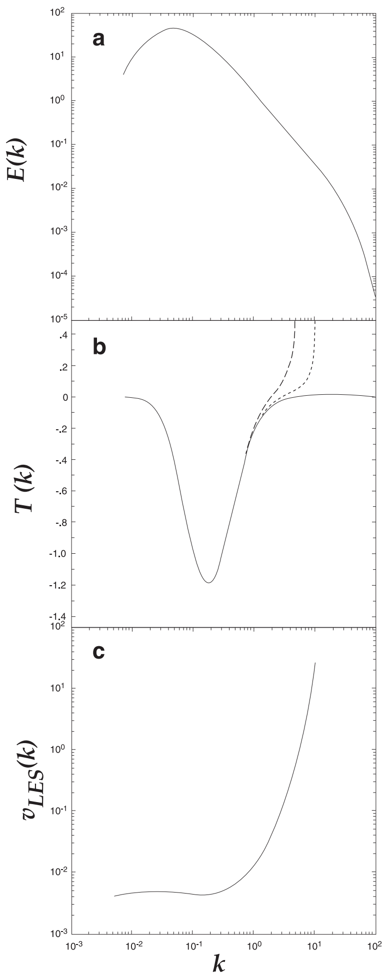

Figure 1. Energy Spectrum $E(k)$, (a); energy transfer spectrum, $T(k)$, (b); and LES eddy viscosity $\nu_{LES}(k)$; (c), for decaying turbulence, according to (18), (19), and (21). In (b) solid line is for $k_c = \infty$, and (c) is for $k_c = 10$ corresponding to the dashed line of (b).

Instead of using the full spectral closure as embodied in a Lagrangian DIA, we employ the simpler TFM (Kraichnan 1971), Herring and Kraichnan 1973). Such a theory furnishes an equation of motion for the energy spectrum, of the form

$$(\partial_t + 2\nu k^2)E(k) = T(k) \tag{18}$$

where $T(k,t)$ is the combination of the nonlinear term as well as the eddy viscosity term in (10). We shall not spell out in full the form of $T(k,t)$ except to note that it is representable in terms of contributions from wave numbers $(\mathbf{p},\mathbf{q})$, $\mathbf{k} = \mathbf{p}+\mathbf{q}$: just as in Boltzmann's equation, there is a "scatter in" term $\sim E(p)E(q)$, and a "scatter out" term $\sim E(k)E(q)$, so that the general form is

$$T(k,t) \sim \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} dp\,dq\,\theta(k,p,q) \{a(k,p,q)E(p)E(q) - b(k,p,q)E(k)E(q)\} \tag{19}$$

The factor $\theta(k,p,q)$ specifies the correlation time for modes $\mathbf{k},\mathbf{p},\mathbf{q}$, as derivable from $G(k,t,t')$ in (8), and its magnitude is indicated by (14). An important point is that the triadic form of T in the theory survives truncation at wave number k_c. Thus, we can write an equation identical to (18), except that contributions from $E(k)$ to T, $k > k_c$ are excluded:

$$(\partial_t - 2\nu k^2)E_c(k) = T_c(k) - [2\nu_{eddy}(k,t)k^2 E_c(k,t)] \tag{20}$$

(Here, the bracketed term will be discussed shortly; at present it is set to zero.) Note that simpler theories such as those discussed in Batchelor's monograph (1953) do not capture this attribute, since the nonlinearity they express is not of a triadic nature. Suppose now we evaluate $T_c(k,t)$ for a high Reynolds number flow, for which $k_{max} \sim k_s \equiv (\varepsilon/\nu^3)^{1/4}$, and various k_c, all $<< k_s$, and for a decay time well into the asymptotic form of $E(k,t)$ and $T(k,t)$ (*ie:* $E_{tot}(t) \sim t^{-p}$). Figure [1] shows the energy spectra, $E(k,t)$, $T(k,t)$, and $T_c(k,t)$, for three values of k_c, and for a moderate Reynolds number flow $R_\lambda \sim 300$. Notice that $T(k,t)$ and $T_c(k,t)$ are closely similar in the energy containing region (the negative lobe of $T(k,t)$), but quickly diverge as $k \to k_c$, with a cusp in $T_c(k,t)$ at $k = k_c$. Of course, if we use (20) to evolve $E_c(k,t)$, we would find that energy piles up just inferior to k_c, and $E_c(k)$ and $E(k)$ would quickly diverge, even in the energy containing region.

However, we may ask if there is a $\nu_{eddy}(k,t)$ such that $E_c(k,t) = E(k,t)$, for $k \le k_c$. Clearly,

$$2\nu_{eddy}k^2 = (T_c(k,t) - T(k,t))/E(k), \tag{21}$$

which is shown in Figure 1b for various k_c. Notice that $\nu_{eddy}(k,t)$ is quite small in the energy containing region, and if k_c is placed near the transition scale k_0, where $T(k_0,t) = 0$, we may propose the following approximate recipe for computing $E(k,t), k \le k_c$: use (20) on ($0 \le k \le_c$) without the ν_{eddy} term. Then use (21) assuming $T(k,t) = 0$. The prescription may be used in direct numerical simulations, provided k_c is successfully large so that k_c is in the inertial range. In such a LES no theory is needed, provided k_c is such that $T(k_c) \sim 0$. Of course, in practice, the range $0 \le k \le k_c$ is quite large ($k_c \sim .2k_s$), but the range $k_c \le k \le k_s$ is vast.

A practical question is how close k_c needs to be to k_0, for the recipe to be accurate. Here we may note that as $R_\lambda \to \infty$ the

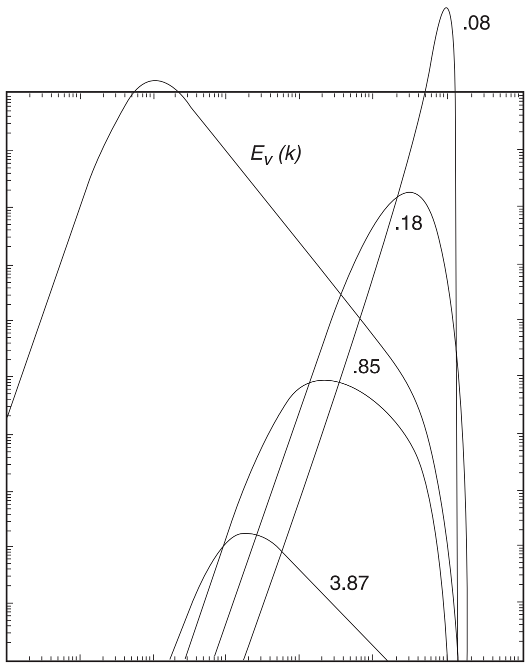

Figure 2. Energy spectrum $E_v(k)$ and scalar variance spectrum $E_\theta(k)$ during decay of $R_\lambda \sim 500$ turbulence. Initial $E_\theta(k)$ is given by (22), and numbers give time (in large–scale eddy turnover units) after injection of the scalar at the Kolmogorov wave number, $k_s = (\varepsilon/\nu^3)^{1/4}$.

minimum of the negative lobe of $T(k)$ does so also, so that an error in k_c need not be too harmful, provided the guess is within the inertia range, where $E(k,t) \sim k^{-5/3}$.

Analytic expressions for $\nu_{eddy}(k)$ have been provided by Kraichnan (1976), who first proposed this spectral form of LES, and by Chollet and Lesieur (1981). The latter authors provide more detailed analytic approximations for both $\nu_{eddy}(k)$ and the equivalent κ_{eddy} to be used at high Reynolds numbers. Results of the spectral LES are summarized in (Lesieur and Lamballais 1999). The results appear to accurately represent the features of flow over a backward facing step, and rotating shear flow.

Again, we stress that this method provides only that the energy spectra is correct, or more generally, that covariance information is correct. This does not assure anything about structural information, which would require accuracy for higher order moments for accurate prediction. Estimates of the correlation $\langle \mathbf{u}(\mathbf{k},t) \cdot \mathbf{u_c}(\mathbf{k},t) \rangle$ show only a modest normalized correlation ($\sim$ 30%) (Herring, 1998). But, such low correlations are typical for other LES methods. Despite these warnings, numerical results using these methods appear impressive, as indicated in the above study of Lesieur and Lamballais.

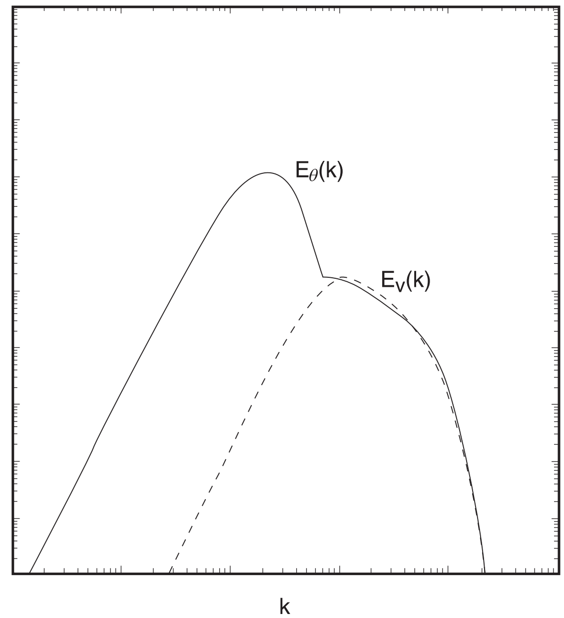

Figure 3. Scalar variance spectrum, E_θ, and kinetic energy spectrum, $E_v(k)$ for moderate R_λ decay. Initial E_θ is given by (23).

PASSIVE SCALAR EVOLUTION IN ISOTROPIC TURBULENCE.

We consider next a passive scalar injected into a developed turbulence. We may vary the injection scale over the complete range of the velocity's scale: from the integral scale, $L_I = (\int_0^\infty E_v(k,t)(dk/k)^{-1}) \sim 1/k_v$, to the dissipation scale $L_d = (\varepsilon/\nu^3)^{-1/4}$. Two–point closure may be employed to study this issue, since we have equations of motion for both $E_v(k,t)$ and $E_\theta(k,t)$. We discuss two extreme cases, the first being that in which an injection of scalar fluctuation is at very small scales ($k_\theta(0) >> k_v(t)$), and the second the converse. For the first,

$$E_\theta(k,0) \sim \delta(k-k_s) \tag{22}$$

Results for this case are shown in Fig. [2] for $R_\lambda \sim 1000$. We note that as time advances, $E_\theta(k,t)$ spreads out in wave number–both to larger and smaller k and the peak wave number, k_θ moves toward the peak velocity wave number, k_v. At the same time, the total scalar variance decreases rapidly. The spread to lower k quickly produces a k^4 "back scatter" region, while beyond k_θ an inertial range $\sim k^{-5/3}$ emerges with time.

Fig. [3] shows the opposite case in which the scalar spectrum initially is centered a much larger scales than that of the velocity. Here

$$E_\theta(k,0) = k^4 \exp(-10k),\ E_v(k,o) = k^4 \exp(-k), R_\lambda(0) \simeq 30. \tag{23}$$

In this case, the velocity field's influence on $E_\theta(k,t)$ is two–fold: in the low–k region, $E_\theta(k,t)$ slowly decays by an eddy–conductivity ($\kappa_{eddy} = 1.1\varepsilon_v^{1/3}L_v^{1/3}$). At higher k $E_\theta(k)$ is entrained by the velocity field in such a way that its spectrum becomes close to that of $E_v(k,t)$, shown here as the dotted line. The evolution of $E_\theta(k,t)$ in both spectral ranges is related to attributes of the closure which we now describe. First, if we examine (10)–(12) for $k \to 0$ for either $E_v(k,t)$ or $E_\theta(k,t)$ (which we jointly denote by $E_{v,\theta}(k,t)$), we note the following equation of motion in the limit $k \to 0$:

$$(\partial_t + 2\nu k^2)E_{v,\theta}(k,t) = C(\{E_v\})k^4 - 2(\nu_{eddy},\kappa_{eddy})k^2E_{v,\theta}(k,t) \quad (24)$$

Here $C(\{E_v\})$ is derivable as the small k limit of the scatter–in term in (19), when written for $E_\theta(k,t)$ [1]. Notice that (24) is suggested on qualitative grounds if there is an equipartitioning amongst independent degrees of freedom at small wave numbers, for which $E(k) \sim k^2$ is a stationary solution to the non–dissipative system. [2]

The closure supports the idea that the ratio, k_θ/k_v (where the k's are the peak wave numbers of E_v and E_θ) tend, at large time to a unique value, which only depends on P_r. This may seem surprising, since if transfer mechanism is local in k, we would expect for $k_\theta/k_v >> 1$, $E_\theta(k,t)$ to be insensitive to k_v. However, as Fig. 2 shows, the "backscatter" region (for which $E_\theta \sim k^4$) is responsible for moving E_θ to ever smaller k, a trend which eventually leads to $k_\theta \sim k_v$. A more detailed analytic study at large R_λ shows the timescale ratio of the velocity field to that of the scalar to be $\sim (k_\theta/k_v)^{2/3}$, as proposed by Corrsin (1955).

Among the early proposals of turbulence theory is the idea that as $P_r \equiv \nu/\kappa$ ranges from ∞ to 0, the inertial range associated with $E_\theta(k)$ ranges from k^{-1} to $k^{5/3}$ to $k^{-17/3}$ (Batchelor, Howells, and Townsend 1959). The closure discussed here conforms to this picture, at large R_λ and P_λ. Here, P_λ is the Peclet number:

$$P_\lambda = \sqrt{E_v}\sqrt{(E_\theta/\int k^2 dk E_\theta)}/\kappa \quad (25)$$

with κ the conductivity. This is shown in Fig. [4], which depicts a series of spectra, $E_\theta(k,t)$ for a range of $P_r \equiv \nu/\kappa$ for decaying isotropic turbulence, $E_v(k,t)$, which is also shown. The computations are taken from Larcheveque *et al.* (1980). We see here the three spectral ranges noted above, with the $E_\theta(k)$ spectrum for high P_r faring over from a $k^{-5/3}$ range to k^{-1} as k approaches the Kolmogorov wave number k_s. We shall not discuss the numerical coefficients of these spectra in any detail, except to note that they are in good agreement with experiments. At low P_r, the data are not in agreement with the experiment of Clay and Gibson (1978), where a k^{-3} range is deduced from Clay's mercury experiment. Moreover, more recent DNS (Gibson *et al.* 1988, 1999) suggest that the scaling of Batchelor *et al.* (1959), noted above, does not obtain and that, in fact, the scaling of the passive scalar is independent of P_r. On the other hand, Chasnov (1992) has examined low P_r regime using a synthetic velocity field, concluding that the Batchelor–Howell–Townsend regime was consistent with his numerics. The Batchelor–Howell–Townsend regime is also indicated in thermal convection numerical studies at $P_r = .07$ (Kerr and Herring 1999).

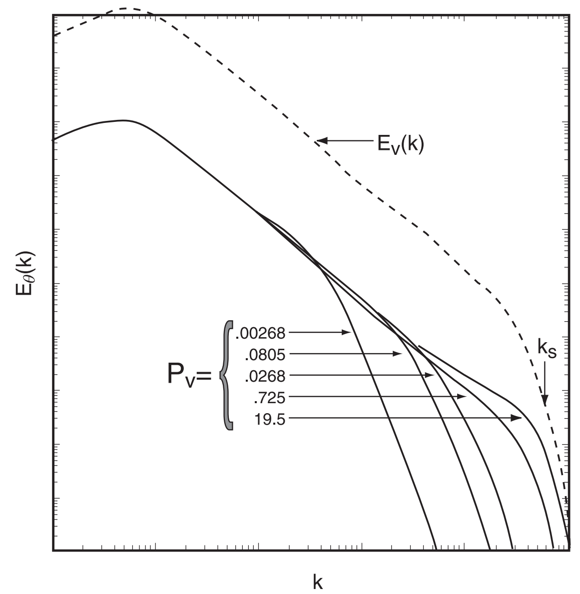

Figure 4. Scalar variance spectra, E_θ for various Prandtl numbers (as marked), and energy spectrum, $E_v(k)$, for $R_\lambda \sim 500$. k_s is the Kolmogorov wave number.

1 TURBULENCE WITH ACTIVE SCALARS: STABLY STRATIFIED FLOW.

Flows that are stably stratified are of great interest in meteorology and even astrophysics. To study such flows, we use the Boussinesq set:

$$\{\partial_t - \sigma\nabla^2\}\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \hat{\mathbf{g}}\theta \quad (26)$$

$$\{\partial_t - \nabla^2\}\theta = -N^2 w - \mathbf{u}\cdot\nabla\theta \quad (27)$$

[1] For the scalar problem, a(k,p,q) and b(k,p,q) are slightly changed, and the contribution of each triad (k,p,q) to T_θ is $\{a_\theta(k,p,q)E_v(q,t)E_\theta(p) - b_\theta(k,p,q)E_v(q)E_\theta(k)\}$.

[2] Recently, Herr *et al.* (1998) have extended the closure to the case of a passive scalar with a uniform mean gradient. Their results indicate satisfactory results as compared to DNS.

Here, θ is the fluctuation of the temperature field about its horizontal mean, and N is the Brunt Väisälä frequency, $N = \sqrt{g\alpha(\partial\overline{T}/\partial z)/T_0}$. In (26)–(28) We have adapted a convenient non–dimensionalization in which the unit of time is L^2/κ, with κ the thermometric diffusivity, and L an arbitrary length scale. The velocity is then in units of κ/L, where L is an arbitrary length scale. Notice that the frequency of oscillation induced by the linearized version of (26)–(28) is:

$$\omega = \sqrt{N^2 sin^2(\vartheta) + 4\Omega^2 \cos^2(\vartheta)} \tag{28}$$

where we omit the viscous and conductive contributions for the moment. We first ask what modifications of the high Reynolds number form of energy spectrum, $E(k,t) \sim \varepsilon^{2/3}k^{-5/3}$ is induced by the additional terms proportional to N and Ω. To do this, we invoke the scaling analysis provided by the general statistical framework discussed at the end of the introduction. On the simplest order–of–magnitude level, we must modify the length of time the strain acts from

$$\theta \sim \frac{1}{\sqrt{k^3E(k)}} \tag{29}$$

to

$$\sim \frac{1}{\sqrt{k^3E(k)} + \hat{\omega}} \tag{30}$$

where $\hat{\omega}$ represents an appropriate angular average of ω as given by (29). We are assuming the flow to be roughly isotropic in the inertial range. Carrying through the identification of ε with the integral of the transfer function, we obtain (in order of magnitude):

$$E(k) \sim \sqrt{\varepsilon\hat{\omega}}k^{-2} \tag{31}$$

Finally, we may estimate the decay of total energy by the same arguments as at the conclusion of Sec. 1. For $\hat{\omega} \to \infty$ we find:

$$E_{tot}(t) \sim t^{-5/7} \tag{32}$$

Our estimates of this section are admittedly crude, and are only slightly more that dimensional analysis. In particular, they make no statement about the degree of anisotropy, and its effects on the energy and temperature spectrum. However, for the problem of rotating turbulence, Yeung and Zhou (1996) confirm (31) through numerical simulation. We should also note that the form of (31) is also indicated by the weak–wave turbulence theory of Caillol and Zeitlin (1999). The latter calculation is restricted to the wave component, but in this context the anisotropy as a function of scale is derived.

The study of the complete theory (DIA) for stratified turbulence has been undertaken by Sanderson (1995), with results indicated in Sanderson *et al.* (1996). A similar detailed study of stratified turbulence using the EDQNM theory is to be found in Godeferd and Cambon (1994). These studies compare well with equivalent DNS. How well they fair in predicting large R_λ flows as encountered in geophysical context remains to be seen.

2 CONCLUDING REMARKS

Our brief survey of the statistical theory of turbulence began with a sketch of its derivation, followed by applications to a variety of flow circumstances, which illustrates both its virtues and failures. The derivation sketched in the introduction stresses an underlying implicit near–Gaussian assumption of the theory, and the necessity of a generalized eddy viscosity (or conductivity). The method of derivation sketched here makes clear that no arbitrary constants appear in the eddy viscosity. The near Gaussian assumption seems equivalent to regarding the flow as nearly structureless. Thus, we should avoid applications to problems in which isolated structures are known to dominate.

We have avoided discussing applications to inhomogeneous flows since such would be unwieldy and cumbersome without necessarily illustrating any new physics. Moreover, the straight–forward application of the theory to such real flows presents formidable numerical problems. It is clear that serious consideration should be given to devise a strategy to reduce the computational load to a manageable level, along the lines of second order modeling, and advocated, for example, by Yoshizawa. The problem here is that it is difficult to ask researchers to invest heavily in such an effort, when the resulting reduced theory may be inaccurate because the underlying theory lacks the ability to cope with structural effects. In this circumstance, a useful strategy is simply to extract simple results from the theory, such as those presented in our applications for their possible insights into the mechanics of turbulence.

On the other hand, inhomogeneous problems generally have rapid distortion (quasi–linear) effects, which are known to represent a significant part of the overall dynamics, and these are treated exactly by the statistical theory. Hence, the application of the full statistical theory to stably stratified flows (Sanderson *et al.* 1993), a flow in which the rapid distortion terms are known to play a dominant role (Hunt and Carruthers 1990), (Kaneda and ?? (1998). If the rapid distortion terms are sufficiently dominant, it is possible that the overall results of the closure would be accurate despite its crude treatment of structures, especially if structures reside at small scales. In fact, recent developments that extend RDT to a joint physical and wave number space suggest that an LES based on such an extended RDT results in a greatly improved theory (Larval *et al.* 1999).

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