Some Analytic Results in the Theory of Thermal Convection

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ABSTRACT

An analytic approximation method is developed to treat the problem of thermal convection at large Rayleigh numbers R. The method is applied to a convection model in which the fluctuating self-interactions are omitted. The results of the method compare satisfactorily to the exact solutions at large Rayleigh numbers. The results include a derivation of the $R^{1/3}$ law for the Nusselt number, and closed form estimates for the shape of the mean temperature field and temperature and velocity fluctuation fields.

1. Introduction

In two previous papers (Herring, 1963, 1964) a method was presented for treating turbulent thermal convection. The procedure consists in deleting from the Navier-Stokes equations those non-linear terms which represent the interaction of the turbulence with itself, i.e., the fluctuating self-interactions. The non-linear terms responsible for the maintenance of a statistically steady state are retained by the model. The model system was proposed as a way to approximate some of the statistical features of turbulent convection, and was not intended to represent an approximate treatment of the amplitude equations for the velocity and temperature fields.

The method presented in the two previous papers is closely related to the Malkus (1954) theory of thermal convection, and especially to a reformulation of his theory by Spiegel (1962). The Malkus treatment seeks an upper bound to the heat transport, subject to the constraints that 1) the horizontal average of the temperature gradient in the fluid be everywhere negative, and 2) there shall be a "smallest scale of motion" contributing to the convective heat transport.

The smallest scale of motion was determined by a stability argument, i.e., it was supposed to be the smallest scale which would grow on the mean vertical temperature field. In an effort to make this smallest scale of motion more exact, Spiegel was led to investigate the equations upon which the present method is based. In the Spiegel treatment, these equations serve as a way of generating a convenient normal-mode expansion set. In terms of this set the Malkus theory approximates the convective heat transport by a finite sum of terms which are bilinear in the velocity and temperature eigen-modes. The member of this set which has the smallest positive growth rate then represents the smallest scale of motion.

The present method, on the other hand, uses these equations as a complete statistical system with its own stability properties. The relationship between the present method and that of Spiegel and Malkus is discussed at greater length by Howard (in press).

In the numerical procedure used by Herring (1963, 1964), the system was allowed to evolve in time until a statistically steady state was obtained. Such a procedure, however, is not suitable for obtaining results at very large Rayleigh numbers ($R > 10^6$) and methods developed here are suitable to these cases also.

The present paper presents analytic and asymptotic results for the convection model in which the fluctuating self-interactions are omitted. The analytic method proposed here takes advantage of the strong coupling of the small-scale temperature field to the dominant large-scale fluid motion. The approximate system including only these interactions is solved exactly, and the coupling of the other scales of motion is then introduced as a correction.

The analytic method is accurate at all values of R provided the horizontal wave number which supports the convection is not too large. The method permits accurate estimates of the shape of the mean temperature as well as the field temperature and velocity fluctuation fields at very large R.

The methods of the present paper are best suited to the case of free boundaries. The use of rigid boundaries complicates the dynamics. At large Rayleigh numbers, and for free boundaries, the temperature fluctuation field receives its main input from the normal velocity gradient at the boundaries. For rigid boundaries, the normal gradient vanishes so that an analysis of the rigid boundary problem becomes more involved. However, we may still use the iteration procedure for rigid boundaries. Some results for the rigid boundary and for $R > 10^6$, obtained in this way, are discussed briefly in Section 5.

2. Equations of motion

The steady state equations of motion for the non-fluctuating system are:

$$\nabla^4 w = -R\nabla_1^2\theta, \tag{1a}$$

$$\nabla^2\theta = -\beta(z)w, \tag{2a}$$

$$\frac{\partial\beta}{\partial z} = -\frac{\partial}{\partial z}(\overline{w\theta}). \tag{3a}$$

These equations are derived from a non-dimensional form of the Boussinesq approximation to the Navier-Stokes equation by deleting those non-linear terms having the form of the deviation of a bi-linear quantity from its horizontal average. The discussion of the theoretical significance of this procedure and a derivation of equations is given in Herring (1963). Here, $w(\vec{r})$ is the vertical component of the velocity field, θ is the temperature fluctuation field, $\beta(z)$ is the horizontally averaged temperature gradient and R is the Rayleigh number. To investigate the solutions to (1), (2) and (3), we analyze the fields into Fourier components[1] as follows:

$$w(\vec{r}) = \pi\sqrt{2}\sum_{n=1}^{\infty} w_n e^{i\pi\vec{\alpha}\cdot\vec{X}} \sin n\pi z, \tag{4}$$

$$\theta(\vec{r}) = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \theta_n e^{i\pi\vec{\alpha}\cdot\vec{X}} \sin n\pi z, \tag{5}$$

$$\beta(z) = 1 + \sum_{n=1}^{\infty} \bar{\beta}n \cos 2n\pi z. \tag{6}$$

Substituting (4), (5) and (6) into (1a), (2a) and (3a) gives

$$w_n = J_n\theta_n, \tag{1b}$$

$$[(2n-1)^2+\alpha^2]\theta_{2n-1} = \tfrac{1}{2}\Sigma(\bar{\beta}_{|n-p|} - \bar{\beta}_{n+p-1})w_{2p-1}, \tag{2b}$$

$$\bar{\beta}_n = -\Sigma w_{2p-1}[\theta_2(n+p) - 1 + \sigma(2p-2n-1)\theta_{|2n-2p+1|}], \tag{3b}$$

where

$$J_n \equiv R\alpha^2/\pi^4(n^2+\alpha^2)^2,$$

$$\beta_0 \equiv 2,$$

and

$$\sigma(X) \equiv 1, \quad X>0$$
$$= 0, \quad X=0$$
$$= -1, \quad X<0.$$

[1] The use of Fourier series affords no particular advantage in formulating an analytic method for treating the present system. We use it only to facilitate a comparision of the method to the numerical results. For a treatment of the problem in terms of spatial variables, see Orszag, S. A., 1964: Notes of summer study program in geophysical fluid dynamics of the Woods Hole Oceanographic Institution, Vol. II. Unpublished, p. 99 *et seq.*

The conservation of heat flux is obtained from (3b) by summing over n, i.e.,

$$N = \sum_1^{\infty} w_n\theta_n + 1, \tag{7}$$

$$= \sum_1^{\infty} \bar{\beta}_n + 1. \tag{8}$$

Here N is the Nusselt number. In writing (2b) and (3b), advantage has been taken of the symmetry of the problem, and only those amplitudes which are non-vanishing in the stable steady state are recorded. Note that in Eq. (5), $\theta(\vec{r})$ contains only one horizontal wave number $\vec{\alpha}$. This is not a restriction for free boundaries in view of the fact that the stability analysis indicates that the stable system contains only a single $|\vec{\alpha}|$.

From Eq. (1b) and the definition of $J_n(\alpha)$, we observe that if $\alpha = 0(1)$, w_n will decrease much more rapidly than θ_n. This leads us to expect that the sums on the right hand side of (2b) and (2c) may be simplified by putting $w_n = 0$ for $n \gg 1$. In the next section we investigate the simple approximation obtained by putting $w_n = 0$ for $n > 1$ in Eqs. (2b) and (2c). The approximation is shown to give accurate results for $\alpha = O(1)$, and arbitrary R. In Section 4 the approximation is improved by reintroducing the higher velocity modes in a systematic way. In the next two sections, the analysis is restricted to free boundary conditions.

3. A simplified model system

If $J_n = 0$, Eqs. (2b) and (3b) simplify to the following:

$$\theta_{2n-1} = \tfrac{1}{2}J_1\theta_1 T_n(\bar{\beta}_{n-1} - \bar{\beta}_n), \quad n>1, \tag{9a}$$

$$\beta_n = w_1(\theta_{2n-1} - \theta_{2n+1}), \tag{9b}$$

where

$$T_n = \frac{1}{(2n-1)^2+\alpha^2}.$$

Note that for $n=1$, (9a) becomes an equation for β_1, i.e.,

$$\beta_1 = 2[1 - R_0(\alpha)/R] \tag{9c}$$

where

$$R_0(\alpha) = \pi^4\frac{(1+\alpha^2)^3}{\alpha^2}.$$

To make the above system determinate we supplement it with the approximate form ($J_n = 0$ for $n>1$) of the heat flux equation (7).

To solve this system, we introduce (9a) into (9b) to get the following difference equation for β_n, i.e.,

$$T_{n+1}\beta_{n+1} + T_n\beta_{n-1} - \left(\frac{2}{J_1(N-1)} + T_n + T_{n+1}\right)\beta_n = 0. \tag{10a}$$

Eqs. (8), (9c) and (10a) comprise a complete set for β_n. Once β is known, θ_n may be most simply found from (9b), summed from n to ∞, i.e.,

$$\theta_{2n-1}=\theta_1 \sum_{p=n} \beta_p/(\mathrm{N}-1). \tag{10b}$$

Eq. (10b) is obtained from (9b) by summing it over n and using (7).

Eqs. (10) may be conveniently solved numerically by iteration, by assuming a trial value for N, computing β_n, recomputing N from (8), etc., until convergence is achieved. In practice the procedure is rapidly convergent. However, if N is fairly large ($\gtrsim 2$), a simpler approximate procedure becomes accurate. If N is large, the β_n becomes a smoothly decreasing function of b. Eqs. (10) may then be replaced by a suitable differential equation with little loss of accuracy. To this end we define smooth functions of $\beta(n)$ and $T(n)$ by the equations

$$\frac{d\beta(n)}{dn}=\tfrac{1}{2}(\bar{\beta}_{n+1}-\bar{\beta}_{n-1}),$$

$$\frac{d^2\beta(n)}{dn^2}=\bar{\beta}_{n+1}+\bar{\beta}_{n-1}-2\bar{\beta}_n,$$

and

$$T(n)\cong T_{n+\frac{1}{2}}.$$

Introducing these definitions into (10a) gives for $\beta(n)$

$$\frac{d}{dn}\left\{\left(\frac{1}{4n^2+\alpha^2}\right)\frac{d\bar{\beta}}{dn}\right\}-\frac{2\bar{\beta}(n)}{J_1(\mathrm{N}-1)}=0. \tag{11}$$

To solve (11), we put

$$\bar{\beta}(\zeta)=\int_\zeta^\infty (\tfrac{1}{4}\zeta'^2+\epsilon)f(\zeta')d\zeta',$$

where $\zeta=2n[2/J_1(\mathrm{N}-1)]^{\frac{1}{4}}$, $\zeta_0=\zeta(n=1)$ and $\epsilon=\zeta_0^2\alpha^2/16$. Substituting this form for $\bar{\beta}(\zeta)$ into (11) gives

$$\frac{d^2f}{d\zeta^2}-(\tfrac{1}{4}\zeta^2+\epsilon)f(\epsilon)=0. \tag{12}$$

Eq. (12) has for solutions the parabolic cylinder functions, $D_{-\frac{1}{2}-\epsilon}(\zeta)$, as defined by Magnus and Oberetinger (1949). These are conveniently represented by a definite integral

$$D_{-\frac{1}{2}-\epsilon}(\zeta)=\frac{1}{\Gamma(\frac{1}{2}+\epsilon)}\int_0^\infty e^{-\zeta t-\frac{1}{2}t^2}t^{\epsilon-\frac{1}{2}}dt. \tag{13}$$

According to (12) $\beta(\zeta)$ may then be written as

$$\bar{\beta}(\zeta)=\bar{\beta}_1\frac{1}{D'(\zeta_0)}D'(\zeta),$$

and hence,

$$N-1\simeq\int_1^\infty dn\beta(n)=\frac{D(\zeta_0)}{D'(\zeta_0)\zeta_0}. \tag{14}$$

Here the primes denote differentation with respect to ζ. The corresponding spectrum for θ_n may be worked out from (10b) i.e.,

$$\theta_{2n-1}=\theta_1\frac{D(\zeta)}{D(\zeta_0)}. \tag{15}$$

We may now extract N as a function of R from (14) by using the definition of ζ_0, and by assuming that N is large enough so that the argument of the D-functions in (15) may be replaced by zero. The result is

$$\mathrm{N}=1+\frac{1}{2}\left\{\frac{\Gamma(1/4+\epsilon/2)}{\Gamma(3/4+\epsilon/2)}\right\}^{\frac{4}{3}}\left(1-\frac{\mathrm{R}_0(\alpha)}{\mathrm{R}}\right)^{\frac{1}{3}}\left(\frac{\mathrm{R}\alpha^2/\pi^4}{(1+\alpha^2)^2}\right)^{\frac{1}{3}}. \tag{16}$$

Here, ϵ is given just before Eq. (12).

For $\alpha=1$, Eq. (16) gives $\mathrm{N}=1+0.27\mathrm{R}^{\frac{1}{3}}$, whereas for the numerical solution, $\mathrm{N}=1+0.31\mathrm{R}^{\frac{1}{3}}$. For large α, however, Eq. (16) is not so accurate. The source of this inaccuracy is not hard to find. For large α, the sequence $J_n(\alpha)$ decreases more slowly than for $\alpha=O(1)$. Consequently, the assumption that $w_n(n>1)$ makes no contribution to the right hand sides of (2b), and (3b) becomes rather poor. In the next section we show how this defect may be remedied in a simple way.

4. An improved approximation

We now improve the method by systematically introducing the higher velocity modes into the dynamics. To do this, observe first that the simplified system, (9a) and (9b), become the exact equations for β_n and θ_n provided T_n in (9a) is changed to $T_n\cdot\tau_n$, where

$$\tau_n\equiv\sum_{p=1}^\infty\frac{(\bar{\beta}_{|n-p|}-\bar{\beta}_{n+p-1})}{\beta_{n-1}-\beta_n}\cdot\frac{J_{2p-1}\theta_{2p-1}}{J_1\theta_1}, \tag{17a}$$

and β_n in (9b) is replaced by $\beta_n\cdot\rho_n$, where

$$\rho_n\equiv\frac{J_1\theta_1(\theta_{2n-1}-\theta_{2n+1})}{\sum_1^\infty J_{2p-1}(\theta_{|2n-2p+1|}\sigma(2n-2p+1)-\theta_{2n+2p-1})\theta_{2p-1}}. \tag{17b}$$

Going through the same steps that led to (10a) and (10b) now gives

$$T_{n+1}\beta_{n+1}+T_n\beta_{n-1}-\left(\frac{2\rho_nG}{J_1(\mathrm{N}-1)}+T_n+T_{n+1}\right)\beta_n=0, \tag{18a}$$

and

$$\theta_n = \theta_1 G \sum_{p=n}^{\infty} \rho_p \beta_p / (N-1), \tag{18b}$$

where

$$G = \sum_{1}^{\infty} w_n \theta_n / w_1 \theta_1. \tag{18c}$$

The exact solution may now be obtained numerically by iterating (18) in the same manner as described in Section 3. For $R \to \infty$, we may again introduce the continuous approximation to the θ_n and β_n spectra. If this is done, (17a) and (17b) simplify to

$$\tau_n \to \sum_p (J_{2p-1}/J_1) x_p \cong 1/\rho_n, \tag{19}$$

with

$$x_n = \theta_{2n-1}/\theta_1.$$

In deriving (19), terms like $(\beta_{1n-p1} - \beta_{n+p-1})$ are approximated by $(2p\text{-}1)(\bar{\beta}_{n-1} - \beta_n)$, with a similar approximation[2] for (17b). This approximation is valid to $O(1/N)$, provided N is large.

Now note that (19) gives values for ρ_n and τ_n which are independent of n. Hence, the solution using (19) for ρ and τ may be obtained from the simplified approach of Section 3 by a simple scale change of the independent variable ζ, i.e.,

$$\zeta \to \zeta' = 2[2G/J_1(N-1)\tau^2]^{\frac{1}{4}} n.$$

With this modification the formulas of the last section may be taken over in their entirety to give an improved approximation. Aside from the above scale change, the only formula which is changed is (16), which becomes.

$$N-1 = \frac{1}{2}\left\{\frac{\Gamma(1/4+\epsilon/2)}{\Gamma(3/4+\epsilon/2)}\right\}^{\frac{3}{4}} \left(\frac{R}{\pi^4}\right)^{\frac{1}{3}} \left(\frac{\beta_1}{2}\right)^{\frac{4}{3}} C(\alpha),$$

$$\cong 0.4617\left(1 - \frac{R(\alpha)}{R}\right)^{\frac{4}{3}} R^{\frac{1}{3}} C(\alpha), \tag{20}$$

where

$$C(\alpha) = (\tau^2/G)^{\frac{1}{3}}.$$

This equation (20) actually is asymptotically exact for $N \to \infty$. Of course the x_n's must still be obtained from (18b) to evaluate τ and G. This may be done by iteration. However, if α is not too large, the x_n's will decrease rather slowly and to obtain τ and G they may be approximated by a constant value say, X. The result is

$$X = -\tfrac{1}{2}S + \sqrt{\tfrac{1}{4}S^2 + S}, \tag{21}$$

where

$$1/S = \sum_{1}^{\infty} (2p-2) J_{2p-1}/J.$$

Figs. 1 and 2 gives $C(\alpha)$ and $X(\alpha)$ for a range of α.

The formulas developed here for the Fourier spectra β_n and θ_n may be used to obtain $\theta^\alpha(z)$ and $\beta(z)$. The calculation is given in the Appendix.

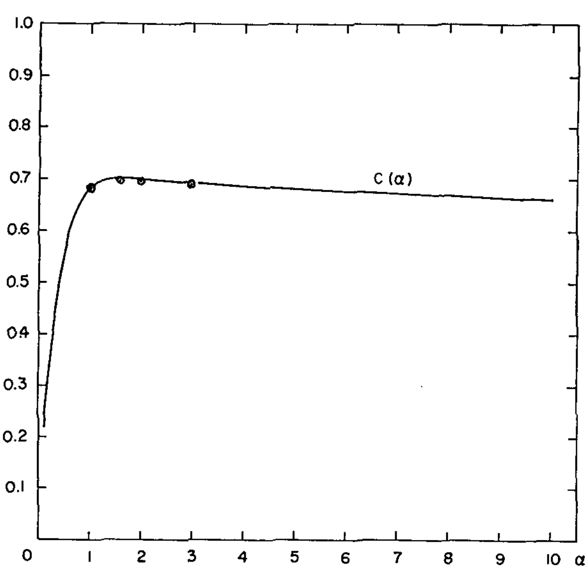

FIG. 1. Comparison of analytic results and numerical calculations for the heat transport shape factor $C(\alpha)$ as a function of α. [Eqs. (20) and (21)].

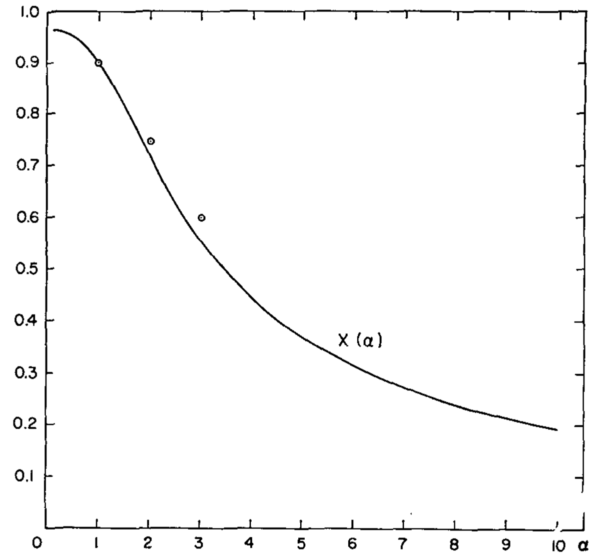

FIG. 2. $X(\alpha)$ as a function of α, according to Eq. (21).

5. Discussion of results

The results of Sections 3 and 4 indicate that for free boundaries the $N \sim R^{\frac{1}{3}}$ law applies to the model which omits the self-interactions, provided $R \gtrsim 10^4$. This result enhances confidence in the physical significance of the present model. The accuracy of the analytic method presented in Section 4 is indicated in Fig. 1, which gives

[2] From (17a) and (17b), it may be shown that (19) becomes accurate for $n \cong N^{\frac{1}{3}}$ provided $\alpha = O(1)$.

a comparison between Eq. (20) and the exact numerical results at $R=10^8$. The circled points were computed numerically by machine methods. The curve represents the analytic results of Eq. (20) for $C(\alpha)=(N-1)/0.4617R^{\frac{1}{3}}$. The latter results were obtained by using the formula of Section 4 in the iteration procedure discussed there. The accuracy of Eq. (20) is good, at least for the moderate values of α for which machine results exist. At large α, (20) implies that the heat transport becomes independent of α, provided $\alpha \gtrsim \alpha_0 = (R/\pi^4)^{\frac{1}{4}}$. For $\alpha > \alpha_0$ the convective heat transport is zero.

Both the machine calculations and the analytic approximation indicate that the value of α which maximizes N has a finite asymptotic value. The value of α for which N is maximum is $\alpha_{\max} \cong 1.55$ for $R=10^8$, while the analytic results give $\alpha_{\max} \cong 1.60$. The numerical results of Herring (1963) at lower Rayleigh number ($R \leq 10^6$) led us to believe that $\alpha_{\max}$ continued to increase as R increased, but apparently this is not the case for free boundaries.

As observed by Herring (1963) the stable solutions of the non-fluctuating system (for free boundaries) are characterized by a single α, α_s, whose value is somewhat smaller than $\alpha_{\max}$. The stability analysis carried out at $R=10^8$ indicates that this is true for $R \rightarrow \infty$ also. Thus, the stable solutions at $R=10^8$ contain only a single $\alpha_s = 0.85$. This value appears to be an accurate asymptotic estimate of α_s. The numerical results at $R=10^8$ give $N(\alpha^s)=142.74$, and $N(\alpha^{\max})=149.93$.

The physical picture of the free boundary convective process predicted by the model is that of a large-scale motion dominating the central region between the conducting plates. This large-scale motion sweeps with it the temperative fluctuation field whose main variations occur in a thin boundary layer of vertical extent $\sim 1/N$. The horizontal scale of both the dominant motion and the temperature fluctuation field is comparable to the distance between the conducting plates.

For rigid boundaries the model system predicts quite different results, especially at large R. Using the interaction method of Section 4, we have extended the rigid boundary calculation to $R=10^9$ for single α. In this case, the value of α which maximizes N continues to increase with increasing R in such a way that the horizontal scale of motion is roughly the boundary layer thickness ($\pi\alpha \sim N$). Over a rather wide range of Rayleigh numbers ($3\times10^4 \lesssim R \lesssim 10^8$), the N result number vs. Rayleigh number may be accurately represented by a 1/3−law. However, for $R \gtrsim 10^8$, N increases significantly less rapidly than $R^{1/3}$. Apparently, above $R \sim 10^8$, the asymptotic boundary layer results of Stewartson (Roberts, 1965), which give $N \sim R^{3/10} (\ln R)^{1/5}$ becomes applicable.[3] It should be stressed that these results are for single α, and the stability analysis of the rigid boundary problem indicated that the stable solutions for $R \gtrsim 5\times10^5$ must have more than one α. These multi-α solutions have not yet been investigated.

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APPENDIX

In this section we compute the space functions $\theta^\alpha(z)$, $w^\alpha(z)$ and $\beta(z)$ from the results of Sections 3 and 4 for θ_n and β_n. Their evaluation is carried out in the limit of large R, so that sums may be approximated by integrals. For $\beta(z)$ we have.

$$\beta(z) \cong \int_0^\infty \bar{\beta}(n) \cos 2n\pi z dn,$$

$$= N\left\{1-\eta\int_0^\infty \frac{D(\zeta)}{D(\zeta_0)} \sin\eta\zeta d\zeta\right\}, \qquad \text{(A1)}$$

where

$$\eta = \frac{2\pi}{\xi_0} z = \sqrt{2}\pi \frac{\Gamma(3/4+\epsilon/2)}{\Gamma(1/4+\epsilon/2)} Nz \cong 1.5017 Nz.$$

The second equality in (A1) results from using (14). Let the last term in the brackets of (A1) be called ψ. Then, it follows that ψ satisfies the inhomogeneous equation

$$\frac{d^2\psi}{d\eta^2} - 4(\eta^2+\epsilon)\psi = -4\eta,$$

whose asymptotic solution is

$$\psi = \frac{1}{\eta} - \frac{\epsilon}{\eta^3} + \frac{1/2+\epsilon^2}{\eta^5} + \cdots, \quad \text{for} \quad \eta \gg \epsilon, \quad \text{and} \quad \eta \gg 1.$$

Then, for $\beta(z)$, we find

$$\beta(z) = N\left\{\frac{\epsilon}{\eta^2} - \frac{1/2+\epsilon^2}{\eta^4} + \cdots\right\} \cong 0.4852 \frac{\alpha^2}{N^3 z^2} + \cdots. \qquad \text{(A2)}$$

Expressions for $w^\alpha(z)$ and $\theta^\alpha(z)$ may be similarly found, i.e.,

$$w(z) \cong 4\pi^2 (N-1)^2 \left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^2 z + \cdots \cong 3.1894 N^2 z, \qquad \text{(A3)}$$

and

$$\theta(z) \cong (N-1)/W(z) + \cdots \cong \frac{0.3135}{Nz}. \qquad \text{(A4)}$$

[3] A boundary layer analysis of the present problem has also been given by Howard, L. N., 1965: Notes of summer study program in geophysical fluid dynamics of the Woods Hole Oceanographic Institution. Vol. I. Unpublished, p. 124 *et seq.*

Eqs. (A2) and (A4) are valid for $z>1/N$, while Eq. (A3) is valid for $0\leqslant z\ll\frac{1}{2}$, provided, of course, N is large.

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