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Stochastic Modeling of Turbulent Flows

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Abstract

We discuss applications of the statistical theory of turbulence to problems related to oceanographic flows. We begin by considering inviscid flows, for which there is an exact formalism for the underlying probability distribution function for the vorticity field (a solution to Liouville's equation). Such a solution, although remote from real flows, has been used to gain insights into certain features of the latter. We review some of these results, in particular the case of barotropic flow over a random topographic relief, noting briefly some successes and failures. Our remarks here are confined largely to the mean flow in a closed basin. We then describe stochastic models in which the nonlinear terms are modeled as a random stirring supplemented by eddy viscosity, focusing attention on the case of homogeneous flows with no mean circulation. Two methods are discussed: (1) Langevin modeling, in which the turbulence is activated by a white–noise random force, and (2) a modeling in which the temporal aspects of the force are allowed to be determined self– consistently, according to the context of the problem. Here, the random force represents the model's rendering of pressure and advective nonlinearities in the equations of motion. From the perspective of the Langevin model, we note how such theories make contact with simpler heuristic ideas of energy transfer, and discuss how they yield inertial range distribution of eddy scales–sizes. We then apply the second modeling to flow over random topography, noting how part of the vorticity field is locked to the topographic relief, with no temporal fluctuations, in the stationary state. We stress that both types of modeling prescribe generalized eddy viscosities, whose forms are determined self–consistently in terms of the flow's energy spectrum and molecular–dissipation properties. Finally, we note the failure of this method to properly incorporate isolated, intense, vortex structures.

1. Introduction

Ocean currents and eddies are clearly turbulent; thus, a stastical description of their scale–size distribution and associated transport properties may be sought in terms of the language and concepts of non–equilibrium statistical mechanics. The latter discipline finds its best known representative in the renormalized perturbation techniques leading to the Direct Interaction Approximation of Kraichnan (1959), and its Lagrangian extensions (Kraichnan (1964), Kaneda (1981)). Such a point of view naturally has its drawbacks. First, only simple averages (ensemble means, denoted here by $\langle\cdot\rangle$) are easily obtained from such theories. Secondly, the theoretical foundations of these theories seem as yet rather fragile, in that the existence of intense structures is at present beyond their pale (McWilliams, (1984), Herring and McWilliams (1985)). Nonetheless, stochastic models have elucidated certain dynamics of large–scale flows (such as inverse cascading aspects of quasi –geostrophic flows (Kraichnan (1967), and ocean basin guyers (Griffa and Salmon (1987), Holloway (1992)). Their use of computer resources is modest in comparison with ocean modeling *via* direct numerical simulations. Moreover, such methods also offer promise of providing a logical basis for large eddy simulations (LES) (Kraichnan (1976), Chollet and Lesieur (1982), Sadourny and Basdevant (1985)), a method in which the effects of scales beyond the resolution capacity of the computer are represented statistically in terms of well–defined eddy viscosity and conductivity concepts. In this paper, we will review some oceanographic research of which the starting point is the statistical theory of fluid flows. We start with some general underlying concepts, the most fundamental of which is the probability density function, $\mathcal{P}(\psi_1, \psi_2, \cdots, \psi_n, \cdots, t)$ that the dynamical degrees of freedom, $\{\psi_1, \psi_2, \cdots, \psi_n, \cdots\}$ attain values specified by the arguments of $\mathcal{P}$. Here, the degrees of freedom, ψ_n, may be thought of as a collection of velocity and temperature fields, $(\mathbf{u}(\mathbf{x}), T(\mathbf{x}))$, and the index n is a labeling of the set of points $\mathbf{x}$.

Given the equations of motion for $\psi(t)$, which we write as

$$d\psi_i/dt = F_i(\psi_1, \psi_2, \cdots, \psi_n, \cdots, t) \tag{1.1}$$

we may infer for $\mathcal{P}$ the evolution equation (Edwards, (1964)),

$$\partial\mathcal{P}/\partial t = \sum_i \partial\{F_i \mathcal{P}\}/\partial\psi_i \tag{1.2}$$

Notice that in (1.2), ψ_i is a degree of freedom, and is not a function of time, as is ψ_i in (1.1). Technically, we should use different symbols here, but usually the context will suffice to make clear which interpretation is intended (we shall use $\hat{\psi}_i(t)$ to indicate solutions to (1.1) if there seems a danger of confusion.) In fact, the $\psi(t)$ in (1.1) are just the characteristics of (1.2), in the sense that a solution of (1.2) is,

$$\mathcal{P} = \langle \prod_i \delta(\psi_i - \hat{\psi}_i(t)) \rangle_{t=0} \tag{1.3}$$

The notation $\langle\cdot\rangle_{t=0}$ means an average over the ensemble of initial conditions for $\hat{\psi}_i$. The forcing function F_i may be further separated into viscous, non–linear, and stochastic components:

$$F_i = -\sum_j \nu_{ij}\psi_j + f_i + \mathcal{F}_i(t) \tag{1.4}$$

where the first term represents viscous or conductive dissipation (*ie*:$\nabla^2\psi$), $f_i(\{\psi\})$ the non–linear force, and $\mathcal{F}_i$ is a stochastic forcing, which, for the moment, we take as white noise. In case the timescale of $\mathcal{F}_i$ is much shorter than the dynamical timescale, we have the theorem of Novikov (1963):

$$\partial\langle\mathcal{P}\rangle_{\mathcal{F}} = \rho\partial^2\langle\mathcal{P}\rangle_{\mathcal{F}}/\partial\psi^2 + \sum_i \partial F'_i \langle\mathcal{P}\rangle_{\mathcal{F}}/\partial\psi_i \tag{1.5}$$

where

$$\rho = \int_{-\infty}^{\infty} ds \langle\mathcal{F}(t)\mathcal{F}(s)\rangle_{\mathcal{F}}, \quad F'_i = -\sum_j \nu_{ij}\psi_j + f_i \tag{1.6}$$

and we assume the random forcing is stationary. Notice that if the viscous force is of the form $\nu_n \psi_i$, and if we suppress f_i, then $\langle \mathcal{P} \rangle_{\mathcal{F}}$ becomes Gaussian.

At this point, we record the form of f_i

$$f_i = \sum_{jk} C_{ijk} \psi_j \psi_k \tag{1.7}$$

that is associated with advection and pressure effects. For example, for advection of the vorticity, $\zeta(\mathbf{x}, t)$, f_i may be written (perhaps somewhat awkwardly) as:

$$f(\mathbf{x}) = -\int_{\mathcal{D}} d\mathbf{x}' d\mathbf{x}'' \mathbf{u}(\mathbf{x}') \zeta(\mathbf{x}'') \nabla_{\mathbf{x}''} \delta(\mathbf{x}'' - \mathbf{x}') \delta(\mathbf{x}' - \mathbf{x}) \tag{1.8}$$

$$= \mathbf{u}(\mathbf{x}) \cdot \nabla \zeta(\mathbf{x}) \tag{1.9}$$

Finally, we note that for simple two–dimensional turbulence, for which both kinetic energy and vorticity are conserved, we may use the relationship between $\mathbf{u}$ and ζ to write the inviscid equations for $\zeta(\mathbf{x}, t)$ as:

$$\partial \zeta(\mathbf{x}, t)/\partial t = -\mathbf{u} \cdot \nabla \zeta = \int d\mathbf{x} d\mathbf{x}' C(\mathbf{x}, \mathbf{x}', \mathbf{x}'') \zeta(\mathbf{x}') \zeta(\mathbf{x}'') \tag{1.10}$$

where,

$$C(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = \{-\partial_{y'} \partial_{x''} + \partial_{x'} \partial_{y''}\} G(\mathbf{x} \mid \mathbf{x}') \delta(\mathbf{x} - \mathbf{x}'') \tag{1.11}$$

and

$$\nabla^2 G(\mathbf{x} \mid \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{1.12}$$

Here, (1.11) is a full articulation of (1.7).[1] In general, inviscid fluid dynamics dictates global conservation laws for flows in a finite domain. For incompressible flows, such conservation laws include those quadratic invariants responsible for conservation of total momentum, vorticity, energy, helicity (the volume integral of $\mathbf{u} \cdot \nabla \times \mathbf{u}$), and, in two dimensions, squared vorticity. For flow in a closed domain, the total momentum is zero. For quasi–two dimensional flows of interest here, the helicity vanishes. Thus, such inviscid, integral constraints have the general form:

$$I^1 = \sum_i \mathcal{C}_i \psi_i \tag{1.13}$$

$$I_n^2 \equiv \sum_{ij} \mathcal{A}_{ij}^n \psi_i \psi_j, \; (n = 1, 2) \tag{1.14}$$

[1] The Fourier representations of (1.10), (1.11) and (1.12) are perhaps more familiar. These are obtained by introducing $G(\mathbf{x} \mid \mathbf{x}') = (1/2\pi)^2 \int d\mathbf{p} p^{-2} \exp(i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}'))$, and $\delta(\mathbf{x} - \mathbf{x}') = (1/2\pi)^2 \int d\mathbf{q} \exp(-\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}'))$ into (1.11) and forming from (1.10) the equation of motion for $\zeta(\mathbf{k}) \equiv \int d\mathbf{x} \exp(i\mathbf{k} \cdot \mathbf{x}) \zeta(\mathbf{x})$. There results $d\zeta(\mathbf{k})/dt = \sum_{\mathbf{p},\mathbf{q}} C(\mathbf{k}, \mathbf{p}, \mathbf{q}) \zeta(\mathbf{p}) \zeta(\mathbf{q})$,with $C(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \delta(\mathbf{k} - \mathbf{p} - \mathbf{q})(p_x q_y - p_y q_x)/p^2$.

We call I_1^2 energy, and I_2^2 the enstrophy[2]. Conservation laws such as(1.14) correspond to a set of constraints on the form of f_i:

$$\sum_i \mathcal{C}_i f_i = 0 \tag{1.15}$$

$$\sum_{ij} \mathcal{A}^n_{ij}\psi_j f_i = 0,\ n = (1,2) \tag{1.16}$$

A time–independent solution of (1.2) for a system in which F_i is given by (1.7), with constraints (1.13) and (1.14), is any function, E, whose sole argument is any function of the total vorticity, energy, and enstrophy. We shall be particularly interested in the case in which this function is a linear combination, which we denote,

$$\mathcal{H} = \alpha I^1 + \beta I_2^1 + \gamma I_2^2 \tag{1.17}$$

The proof that $\mathcal{P} = E(\mathcal{H})$ satisfies (1.2) follows from direct substitution. In (1.17), (α, β, γ) are Lagrange multipliers, whose values are determined by the condition that $\int \delta\psi I_n(\{\psi\})$ have preassigned values.

Of particular interest is the case for which the function E is an exponential, the *thermal equilibrium* distribution:

$$\mathcal{P}^{therm}(\{\psi\}) = \mathcal{C}\exp(-\mathcal{H}\{\psi\}) \tag{1.18}$$

Here, $\mathcal{C}$ is a normalizing constant such that $\int \delta\psi\mathcal{P}(\{\psi\}) = 1$, $\delta\psi \equiv \prod_{i=1}^{\infty} d\psi_i$. We remark that (1.18) has another interpretation: given a system with constraints (1.13), the most probable distribution of $\{\psi\}$ is (1.16) (Orszag (1970), (1974)). The formal proof proceeds by extremalizing the entropy, $Z \equiv \ln \int \delta\psi\mathcal{P}^{therm}(\{\psi\})$.

We should stress that thermal equilibrium considerations omit entirely dissipative effects associated with viscosity and conductivity. The latter have the effect of selectively dissipating the small scales, which in themselves may not be of interest. But the dissipation scales induce a cascade of energy from the large, essentially inviscid scales toward small scales, and such a cascade is entirely missing from(1.18). To see this, we form the equation of motion for $d\langle\psi_i^2/dt\rangle$. From (1.1), (1.4), and (1.7), we see that the essential term that transfers energy (ψ_i^2) out of mode i is $\sim \langle\psi_i\psi_j\psi_k\rangle \equiv \mathcal{T}_{ijk}$. But this third–order moment is, according to (1.16), zero, by symmetry. The question then is the relevance of thermal equilibrium issues to real flows.

Two points may be made in this connection. First, any theory of turbulence which proposes a formula for $\mathcal{T}_{ijk}$ in terms of $\langle\psi_i^2\rangle$ (and auxiliary quantities) ought to satisfy (1.18), if dissipative effects are turned off. In this sense, equilibrium is a test of a theory. The second and related point is that in regions of phase space remote from the dissipative degrees of freedom, the distribution of variance (energy and enstrophy) among the degrees of freedom may be expected to be equipartitioned. For example, an ensemble of flows whose

[2] These conservation laws correspond to symmetries of $C(\mathbf{x}, \mathbf{x}', \mathbf{x}'')$: $C(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = -C(\mathbf{x}'', \mathbf{x}', \mathbf{x})$, and $\nabla_{\mathbf{x}}^{-2} C(\mathbf{x}, \mathbf{x}', \mathbf{x}'') = -\nabla_{\mathbf{x}'}^{-2} C(\mathbf{x}', \mathbf{x}, \mathbf{x}'')$.

initial statistics are anisotropic should tend towards isotropy, in the absence of anisotropic forcing, and in regions of scale sizes where dissipative effects are small. It remains to be seen if thermal equilibrium can yield more quantitative information. The answer to this question depends on the system investigated, and in particular, the number, if there are inviscid constraints such as in (1.16). In three dimensions, there is only one such constant, the total energy. This leads to the well–known result for the energy spectrum, $E(k) \sim k^2$ (Lee, (1952)). There would seem to be scant conclusions to be drawn from this, except isotropy. Nevertheless, this profile at small k may figure in a vital way in determining the decay of total energy (Saffman (1967)). For quasigeostrophic flow, the number of quadratic constants is larger (≥ 3, depending on the context), and the quantitative conclusions richer. The basic reason for the more interesting domain in near two–dimensional flows is that the additional constraints dramatically steepen the spectra at small scales, mitigating the ultraviolet catastrophe of $E(k) \sim k^2$. We will discuss two examples of the application of thermal equilibrium flows: two–layer quasigeostrophic flow, and barotropic flow over a topographic relief.

In the first example, the two–layer representation of the quasigeostrophic flow, we will, for simplicity, take the layers of equal depth. Also, for simplicity, we take the ensemble mean fields to be zero. The equations of motion assert that each layer's potential vorticity is conserved following fluid particles. The potential vorticity consists of two parts: the vorticity ($\zeta_i = -(\partial_x^2 + \partial_y^2)\psi_i(x, y)$), with ψ_i the stream function for layer $i = (1, 2)$, and the vortex stretching term, $(-1)^i R^{-2}(\psi_1 - \psi_2)$. Here, R is the (external) Rossby radius of deformation. In discussing this problem, it is useful to discriminate between scales of motion which are larger or smaller that R in their lateral extent. For the present purposes, we estimate the scale of $\psi(x, y)$ as $\ell \equiv \sqrt{\langle \psi^2(x, y)\rangle / \langle (\nabla^2 \psi^2 \rangle}$, where we recall that $\langle \cdot \rangle$ has the effect of averaging over the (x, y) periodic domain. The inviscid system has three quadratic constants of motion: the total energy $\langle (\nabla\psi_1)^2 + (\nabla\psi_2)^2 + R^{-2}(\psi_1 - \psi_2)^2 \rangle$, and the enstrophy of each layer, $\langle \{\nabla^2\psi_i + (-1)^i R^{-2}(\psi_1 - \psi_2)\}^2 \rangle$. The thermal equipartition solution to this problem (Salmon *et al.* (1976)) is nearly barotropic $((\psi_1 = \psi_2)$ at scales larger that the Rossby radius, and is a random, equal–parts mixture of barotropic and baroclinic motion at values of ℓ smaller than R ($\langle \psi_1\psi_2 \rangle \sim 0$). Qualitatively, we may understand this behavior in terms of a steepest descent estimation of the integral of (1.18) needed to evaluate second moments of $\{\psi_1,\ \psi_2\}$. At large scales, $\mathcal{H}$ is minimized by a barotropic stratification, since the potential energy, $\sim (\psi_1 - \psi_2)^2$ is then the dominant contributor to $\mathcal{H}$. At small scales, where the gradients are dominant, $\mathcal{H}$ is minimized by independent but equal stream functions. The transition zone $\ell \simeq R$, where the correlation between layers decreases from near unity to zero requires a computation, which has been made by Salmon *et al.* (1976). Their results seem plausible when compared with direct numerical simulations and observations.

We now discuss in more detail the problem of barotropic flow over an arbitrary topographic relief, $h(\mathbf{x})$. The latter we take as non–distributed with respect to the ensemble of flows (the same $h(\mathbf{x})$ for each member of the ensemble). Let ζ be the potential vorticity for barotropic flow above a topography, whose relief is $h(x, y)$. Then,

$$\partial\zeta(\mathbf{x}, t)/\partial t = -\mathbf{u} \cdot \nabla\{\zeta + h(\mathbf{x})\} \tag{1.19}$$

The three conservation laws associated with (1.19) are:

$$I_1 = \int_{\mathcal{D}} d\mathbf{x}\zeta \tag{1.20}$$

$$I_2 = \int_{\mathcal{D}} d\mathbf{x}(\zeta - h)^2 \tag{1.21}$$

and

$$I_3 = \int_{\mathcal{D}} d\mathbf{x}\psi\zeta \tag{1.22}$$

where $\zeta = -\nabla^2\psi$ is the stream function, and $\mathcal{D}$ is the closed domain of the flow. The thermal equilibrium prescription for computing $\langle\zeta(\mathbf{x})\rangle$ is then, from (1.18):

$$\langle\zeta\rangle = \int \delta\zeta\zeta\exp(-\mathcal{H}) / \int \delta\zeta\exp(-\mathcal{H}) \tag{1.23}$$

where

$$\mathcal{H} = \sum_i B_i\zeta_j + \sum_{ij} A_{ij}\zeta_i\zeta_j, \quad \delta\zeta \equiv \prod_i d\zeta_i \tag{1.24}$$

and,

$$B_i = (\gamma - 2\alpha h_i), \quad A_{ij} = \alpha\delta_{ij} - \beta G_{ij} \tag{1.25}$$

Here, $-G_{ij}$ corresponds to the inverse of the Laplacian ∇^2 in (1.12). Then

$$\langle\zeta_n\rangle = \partial/\partial B_n\{\ln\int\delta\zeta\exp(-B\zeta - \zeta A\zeta)\} \tag{1.26}$$

$$= \partial/\partial B_n \ln(Z). \tag{1.27}$$

To proceed, we introduce a transformation of basis $\zeta = \alpha\xi$, such that $A_{ij}\zeta_i\zeta_j = A_{ij}\alpha_{in}\alpha_{jm}\xi_n\xi_m = \lambda_n\xi_n^2$ (summation convention invoked). Here, λ_n are the eigenvalues of the symmetric matrix A. Then,

$$Z = \int \delta\xi\exp\{-\sum[\sqrt{\lambda_n}\xi_n + C_n/(2\sqrt{\lambda_n})]^2 + \sum C_n^2/(4\lambda_n)\} \tag{1.28}$$

$$= \exp(\sum C_n^2/(2\lambda_n)^2)/\sqrt{\mid A \mid}, \quad C_n = \sum_m \alpha_{nm}B_m \tag{1.29}$$

and,

$$\ln(Z) = \sum(B_i\alpha_{im}B_j\alpha_{jm})/(4\lambda_m) - (1.2)\ln(\mid A \mid) =$$

$$(1/4)(BA^{-1}B) - (1/2)\ln(\mid A \mid) \tag{1.30}$$

Hence,

$$\langle\zeta_n\rangle = (1/4)\{A^{-1}B + BA^{-1}\}_n = (1/2)\{A^{-1}B\}_n \tag{1.31}$$

Interpreting (1.31) in terms of its spatial representation and using (1.25) gives:

$$\langle\zeta\rangle(\mathbf{x}) = (1/2)\nabla^2(\alpha\nabla^2 - \beta)^{-1}\{\gamma - 2\alpha h\} \tag{1.32}$$

so that the stream function satisfies:

$$(\alpha\nabla^2 - \beta)\langle\psi(\mathbf{x})\rangle = \alpha h(\mathbf{x}) - (1/2)\gamma \tag{1.33}$$

Finally, we note that two–point covariances $\langle\zeta(\mathbf{x})\zeta(\mathbf{x}')\rangle \equiv Z(\mathbf{x} \mid \mathbf{x}')$ follow from (1.30) and (1.18) as:

$$\langle\zeta(\mathbf{x})\zeta(\mathbf{x}')\rangle = \partial \ln(Z)/\partial A_{ij} = \mathcal{G}(\mathbf{x} \mid \mathbf{x}') + \langle\zeta(\mathbf{x})\rangle\langle\zeta(\mathbf{x}')\rangle \tag{1.34}$$

where, $\langle\zeta(\mathbf{x})\rangle$ is given by (1.32), and $\mathcal{G}(\mathbf{x} \mid \mathbf{x}')$ satisfies:

$$\nabla^2(\alpha\nabla^2 - \beta)\mathcal{G}(\mathbf{x} \mid \mathbf{x}') = -\delta(\mathbf{x} \mid \mathbf{x}') \tag{1.35}$$

In the homogeneous context ($\langle\psi(\mathbf{x})\rangle = 0$), the stream function power spectrum is,

$$\Psi(\mathbf{k}) = 1/(\alpha k^4 + \beta k^2), \tag{1.36}$$

as proposed by Kraichnan (1975). The power spectrum $\Psi(k)$ seems reasonable, but that for $Z(k)$ becomes unrealistically independent of k as $k \to \infty$.

The form of $\langle\psi(\mathbf{x})\rangle$ predicted by (1.33) for a closed basin seem plausible, and related to other deterministic models. Here, we should mention two studies for which (1.33) plays a pivotal role: that of Griffa and Salmon (1989), and that of Holloway (1992). The first of these is a study of closed basin barotropic circulation on a β–plane. This problem is equivalent to (1.19), if we take

$$h(x, y) = \varpi(y - y_0) \tag{1.37}$$

where, ϖ is the (constant) "beta" term, the northward derivative of the Coriolis force. Griffa and Salmon first demonstrate that a numerically simulated ocean basin flow without friction or wind stress evolves toward a solution of (1.33). In this case, the central region of flow is described by $-\beta\langle\psi(\mathbf{x})\rangle = \alpha\langle h(\mathbf{x})\rangle$, while the $\alpha\nabla^2\langle\psi(\mathbf{x})\rangle$ permits the flow to obey slip boundaries in the coastal region. Such flows are quasi–steady, and are designated by them as Fofonoff flows (although the latter is strictly steady (Fofonoff (1954)). For a wind–driven flow with bottom–drag, the flow is quite similar, provided the wind stress exerts a torque which balances the bottom–drag torque.[2] If the wind opposes the Fofonoff flow, then flow becomes turbulent, with a much smaller mean flow. The connection of the forced–dissipative flow with (1.33) is more tenuous.

The study by Holloway examines the relevance of (1.33) to real ocean circulations, utilizing observed $h(\mathbf{x})$. He proposes a simplified picture, based on ignoring the ∇^2 term in (1.33), and finds the (North Atlantic) circulation so produced (called an unprejudiced circulation) plausible, provided regions of direct forcing (such as the Gulf Stream) are

[2] It can be demonstrated that the equipartitioning solution (1.18) survives in the forced–dissipative case, provided there is a mode–by–mode balance of forcing and dissipation.

avoided. Holloway suggests that (1.33) may be a useful zeroth–order circulation, in the sense that a correct representation of eddy viscosities should impel the system toward (1.33), rather than towards a state of rest.

Equation (1.10) has an infinity of constants of motion, (an integral of any function of the vorticity, ζ), but the formalism developed here only uses linear and quadratic constraints. Recently, methods have been developed that are able to incorporate these additional constraints into the statistical mechanics (Weichman (1993), Miller *et al.* (1992)). Such methods have been applied to Jupiter's Red Spot, and its application to Oceanographic problems would seem a useful project. The structure of the mean field equations for $\langle\psi\rangle$ specifies the mean vorticity as a function of the mean stream function.

2. Langevin Modeling

We now turn to methods which are suitable for real, dissipative flows. Our interest here is in obtaining variance information, and we will focus on simple problems in which the ensemble mean flow is absent, and the flow may be regarded as homogeneous with impunity.[3] The principle item of interest here is to obtain accurate information about energy spectra with reasonable economy in computation. If such methods are found, incorporating flows with (ensemble) mean fields presents only technical problems. Over the past quarter of a century, the Langevin equation has proved a useful basis for generating quantitative information at the covariance level. The basic physical *ansatz* is that the nonlinear force (as in (1.1)–(1.7)) acts, to first order, as a random forcing of ψ_i. This starting point, that f_i is nearly Gaussian, despite its dependence on the dynamical variables ψ_i, would seem credible only if the number of degrees of freedom in (1.7) is large. But such a replacement neglects those higher–order correlations necessary to maintain the conservation laws (1.14): the total variance, $\sum\langle\psi_i^2\rangle$ would increase with time. Thus, if we argue that a Langevin equation is a plausible zeroth–order approximation, we must simultaneously augment the molecular dissipation of (1.1)–(1.4) in such a way that conservation principles are observed by the Langevin system, insofar as possible[4]. Thus, the prescription is to replace

$$d\psi_i/dt = \sum_{jk} C_{ijk}\psi_j\psi_k - \nu_i\psi_i \tag{2.1}$$

with

$$d\psi_i/dt = \sigma(t)\sum_{jk} C_{ijk}\sqrt{\theta_{ijk}}\chi_j\chi_k - \mu_i\psi_i \tag{2.2}$$

Here, χ_i are Gaussian fields contrived to have the same covariances as ψ_i, and $\sigma(t)$ is a white–noise, stationary force, $\langle\sigma(t)\sigma(t')\rangle = \delta(t-t')$. The basic idea in (2.2) is to chose θ_{ijk} and μ_i so that the variances of (2.2), $\Psi_i(t)$, approximate those of (2.1) (which we designate $\hat{\Psi}_i$). To see how this may be effected, we advance (2.1) forward in time from an

[3] For homogeneous flows, the Fourier representation renders $\langle\psi_i\psi_j\rangle = \delta_{ij}\langle\psi_i^2\rangle$, where (i,j) stand for $\mathbf{k}_i$ and $\mathbf{k}_j$. We use this simplification in the remainder of this section.

[4] Langevin models maintain conservation principles only in ensemble mean.

initial state of Gaussian chaos: $\mathcal{P}(\psi_1, \psi_2, \cdots) \sim \exp(-\sum_i a_i \psi_i^2)$. To order dt we find for $\hat{\Psi}_i \equiv \langle \psi_i^2 \rangle$

$$(\frac{1}{2}\partial_t + 2\nu_i)\hat{\Psi}_i(dt) = 2dt \sum_{jk} C_{ijk}^2 \hat{\Psi}_j(0)\hat{\Psi}_k(0) + 4dt \sum_{jk} C_{ijk}C_{jki}\hat{\Psi}_i(0)\hat{\Psi}_k(0) \tag{2.3}$$

while from (2.2) we have (exactly, at any time):

$$(\frac{1}{2}\partial_t + 2\mu_i)\Psi_i(t) = 2\sum_{jk} C_{ijk}^2 \Psi_j(t)\Psi_k(t)\theta_{ijk} \tag{2.4}$$

where we have used the constraint $\langle \chi_i \chi_j \rangle = \delta_{ij}\Psi_i(t)$. Comparing (2.3) to (2.4) we see that requiring $\hat{\Psi}_i = \Psi_i$ implies (to order dt^2)

$$\mu_i = \nu_i - 4dt \sum_{jk} C_{ijk}C_{jik}\Psi_k\theta_{ijk} \tag{2.5}$$

and

$$\theta_{ijk} = dt \tag{2.6}$$

Imagine now that the procedure for evaluating (2.4) carried out to arbitrary order in dt. Through second order this recipe presents no problems. But beyond that, such a perturbation analysis of (2.1) will contain a vast number of terms. Only a sub–group will (at any order) have the form of the right–hand side of (2.4), (with suitable extensions of (2.6)). Another sub–group will have the form of (2.5) times $\Psi_i(t)$. Since at orders higher that dt^2 the perturbation method contains terms *not* of the form (2.4), we must admit that at best a judicious selection of factors θ_{ijk} will make (2.4) correspond to certain well defined perturbation schemes, whose validity must be judged on other grounds.

Finally, note that if the damping term μ_i in (2.4) is to fulfill conservation principles (1.14), it must satisfy,

$$\mu_i = -4\sum_{jk} C_{ijk}C_{jik}\theta_{ijk}\Psi_k \tag{2.7}$$

So far, equation (2.4) contains considerable arbitrariness in that the factor θ_{ijk} is as yet unspecified. This quantity has dimensions of time, and physically is the duration of the interactions among modes (i, j, k).

One approach to determine θ_{ijk} is to assume that covariances are accurately given by some form of renormalized perturbation theory, such as the Direct Interaction Approximation (DIA) (Kraichnan (1959)), and then determine θ by a comparison of (2.4) to such a theory. Of course, the DIA is not of the Langevin form, so the determination of θ is only approximate. To see how this may be done, we note that the DIA may be written in the form (2.4), with θ replaced by Θ, where:

$$\Theta_{ijk} = \int_0^t ds G_i(t \mid s)\{\Psi_j(t \mid s)/\Psi_j(t \mid t)\}\{\Psi_k(t \mid s)/\Psi_k(t \mid t)\} \tag{2.8}$$

Here, $\Psi_i(t \mid s) \equiv \langle \psi_i(t)\psi_i(s)\rangle$, and $G_i(t \mid s)$ is the response of mode $\psi_i(t)$ at a small perturbation at time s. With respect to the "two–time" quantities $\Psi_i(t,t')$, we first note the thermal equilibrium relation:

$$\Psi_i^{therm}(t \mid s) = \Psi_i^{therm}(s \mid s) G_i^{therm}(t \mid s) \tag{2.9}$$

By thermal equilibrium, we mean that the distribution of modes ψ_i satisfies (1.18). Equation (2.9) is known as the "fluctuation dissipation relation, " and is frequently used even for dissipative flows. Its extension to non–stationary flows may be justified by an alternate perturbation theory, (Herring, (1965), (1966)), in which (2.9) emerges as a consistent *a priori* estimate of $\Psi_i(t \mid s)$ given the value of $\Psi(s \mid s)$. In the DIA, $G_i(t \mid s)$ satisfies:

$$G_i(t \mid s) = \delta\Psi_i(t,t')/\delta\Psi_i(t',t'), \tag{2.10}$$

so that (2.9) may be satisfied. The DIA equation for $\Psi_i(t,t')$ is,

$$\partial_t \Psi_i(t,t') = 2C_{ijk}^2 \int_0^{t'} G_i(t',s)\Psi_j(t,s)\Psi_k(t,s)$$

$$-4C_{ijk}C_{jki} \int_{t'}^{t} \Psi_i(s,t')G_j(t,s)\Psi_k(t,s) \tag{2.11}$$

To complete the determination of θ_{ijk}, we parameterize $G(t,s)$ by

$$G_i(t,s) = \exp(-\int_s^t ds' \tilde{\mu}_i(s')) \tag{2.12}$$

and use (2.9) (incorrectly, out of thermal equilibrium). This allows the identification of Θ_{ijk} in (2.8) with θ_{ijk} in (2.2). Using (2.12) and (2.8), we find,

$$\frac{d\theta_{ijk}}{dt} = 1 - (\tilde{\mu}_i + \tilde{\mu}_j + \tilde{\mu}_k)\theta_{ijk} \tag{2.13}$$

The $\tilde{\mu}_i$ may be evaluated by integrating the equation of motion for $G_i(t,t')$ ((2.10) applied to (2.11)) over t, and using (2.12) as an approximation for $G_i(t,t')$. For k larger than the peak in the energy spectrum $E(k) \equiv 2\pi k^3 \Psi(k)$, we find (Kraichnan (1959), (Herring (1975)):

$$\hat{\mu}(k) \sim k\sqrt{\int_0^k dp E(p)} \tag{2.14}$$

Notice that (2.14) implies a dependence of $\hat{\mu}$ on large– scale sweeping: an addition of a *random* uniform translation, $\mathbf{u}_0\delta(\mathbf{k})$ to the velocity comprising $E(k)$ adds $k\sqrt{\langle \mathbf{u}_0^2\rangle}$ to (2.14). Although such dependence on large–scale sweeping seems appropriate for G_i, it is not for Ψ_i (or more precisely, the velocity field belonging to Ψ), since small–scale spectra are unaware of the Galilean frame in which they move. The problem stems from adapting the Eulerian frame to formulating the problem, and can be avoided by a formulation in an entirely Lagrangian frame. The latter may approximately be realized by a modification of the characteristic time–scale $\hat{\mu}(k)$ (that of $G_i(t \mid s)$) to a Lagrangian time–scale:

$$\eta(k) \sim \sqrt{\int_0^k dp\; p^2 E(p)} \tag{2.15}$$

Equation (2.15) should then be used in place of $\hat{\mu}$ to compute θ_{ijk}. Such methods have been used in a number of studies to predict energy spectra for quasigeostrophic flows, and we will review certain of these shortly. Before such a summary, however, we should indicate in general terms what violence the Markov modeling as embodied in (2.2) does to the dynamics of the basic equations. We recall that the first step was to replace the actual nonlinear force by a random stirring. Broadly speaking, the effect of the random forcing is to destroy in the model any effects of structures that Navier–Stokes may possess. In this sense, such models are "structureless," and their deviations from reality may give some indication of the effects of structures on covariance information.

From the historic perspective, stochastic models were developed *after* results of the analytic theories (such as the DIA) were known, at least in their broad outline. One reason for their development was to assure that a particular analytic theory was realizable: they were "proof by construction" that the covariance of an analytical theory is physically realizable, a point of some insecurity among theorists in the early days (1955–1964). Finally, it was realized that the stochastic models could be used to generate perturbation expansions in their own right, with the analytic theory serving as the unperturbed state (Phythian (1969), Kraichnan (1971), Herring and Kraichnan (1972)). Stochastic models may also be used as Monte Carlo simulations in complex flows, for which analytic theories may present equations too formidable for practical application. This approach has not been much used, but may be of computational advantage for certain flows, as has been found by Kaneda (1992).

The system ((2.4), (2.7)), with θ given by (2.15), is called an Eddy Damped Quasi-normal Markovian Approximation (EDQNM), and a systematic review of their application to various problems in quasigeostrophic flows may be found in Lesieur (1990) . Of course, such studies rely heavily on numerical methods for solution, and such may not be a suitable basis for gaining insight into the underlying physics. One useful procedure to reveal the physics is to assume that the energy spectrum is strongly peaked at large scales, and approximate the wavenumber integrals in (2.4) by assuming that the wavenumber belonging to either j or k (but not both) is small [6]. This leads to (see Lesieur (1990), p. 278):

$$\partial E(k)/\partial t = k^{-2}\partial_k\{k^3\partial_k\{kE(k)\eta(k)\}\} - 2\nu k^2 E(k) \tag{2.16}$$

where we recall (2.15) for $\eta(k)$. Thus, the energy (or enstrophy) cascade to small scales is diffusive; energy passes from one scale to adjacent scales. The rate at which this happens,

[6] Note that according to (1.10)–(1.11), $\sum_{jk} C_{ijk}C_{jki}\Psi_j\Psi_k = \int d\mathbf{p}d\mathbf{q}\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\cdot B(\mathbf{k},\mathbf{p},\mathbf{q})\Psi(\mathbf{p})\Psi(\mathbf{q})$. What is proposed is to evaluate this integral by assuming $\mathbf{p}$ to be small, (hence, $\sim \mathbf{q} \sim \mathbf{k}$), and expand $\Psi(\mathbf{k}-\mathbf{p})$ about $\mathbf{p}=\mathbf{0}$, retaining only the lead, non–vanishing terms.

$\eta(k)$, is essentially the *rms* strain. Such a modeling of turbulent transfer was first proposed by Leith (1967), on heuristic grounds. Equation (2.16) contains both the inverse cascade of energy (for the case in which a small–scale forcing term is added to its right–hand side), and forward enstrophy cascades. The former leads to the well–known $k^{-5/3}$ inverse cascade range, while the latter leads to a $k^{-3}(\ln(k/k_i))^{-1/3}$ enstrophy range (Kraichnan (1971)).

These methods may be readily extended to anisotropic turbulence, such as occurs with β–plane flows (Holloway and Hendershott (1977)). If we examine the rate equations for anisotropic turbulence at small scales, using the diffusion expansion described above, a rather simple heuristic emerges (Herring, 1975)): at small scales, k, the level of anisotropy at a given time is determined by a balance between its rate of production by large–scale straining (strain rate times the small–scale isotropic component of the spectrum), and its rate of destruction, by isotropization at scales roughly comparable to k. The isotropizing rate is essentially $\eta(k)$, as given by (2.15), times the small–scale anisotropy. Thus, if $E(k)$ falls off faster that k^{-3}, the small scales retain the anisotropy of the large scales.

3. Non–Markovian Stochastic Models

Although Markovian models are easy to implement either analytically or numerically, they are unable to cope with systems that have disparate time scales, as would be encountered for example, in turbulence coexisting with waves. A simple example is barotropic flow over random topography, for which a component of the vorticity is locked to topography. We discuss here briefly the application of the DIA to this case, starting from the point of view of a non–Markovian model of the DIA, as developed in (Kraichnan (1971), and (Herring and Kraichnan (1972)). We take as the basic dynamical variable the Fourier transform of the vorticity field as given by (1.19), which we write as:

$$\partial\zeta_{\mathbf{k}}/\partial t = F_{\mathbf{k}}(\zeta, h) - \nu k^2 \zeta_{\mathbf{k}} \tag{3.1}$$

where $F_{\mathbf{k}}$ is the Fourier transform of the right–hand side of (1.19), whose Fourier transform is written as:

$$F_{\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{p}+\mathbf{q}} \{C_{\mathbf{k},\mathbf{p},\mathbf{q}}\zeta_{\mathbf{p}}\zeta_{\mathbf{q}} + E_{\mathbf{k},\mathbf{p},\mathbf{q}}\zeta_{\mathbf{p}}h_{\mathbf{q}}\} \tag{3.2}$$

To emphasize the idea that (3.1) couples a fast mode ($\zeta(\mathbf{x}, t)$ with an (infinitely) slower component, $h(\mathbf{x})$ we also write,

$$\partial h_{\mathbf{k}}/\partial t = 0 \tag{3.3}$$

We shall first write down the DIA stochastic model for (3.1) and then justify it. It is

$$\partial\zeta_{\mathbf{k}}/\partial t = -\int_{t'}^{t} ds(\rho_{\mathbf{k}}(t,s)\zeta_{\mathbf{k}}(s) + \hat{\rho}_{\mathbf{k}}(t,s)h_{\mathbf{k}}) + F_{\mathbf{k}}(\chi, h) - \nu k^2\zeta_{\mathbf{k}} \tag{3.4}$$

Here, $h_{\mathbf{k}}$ denotes the Fourier transform of $h(\mathbf{x})$, and $\chi_{\mathbf{k}}$ is a Gaussian variable, as in (2.2). The visco–elastic terms, $\rho_{\mathbf{k}}$ and $\hat{\rho}_{\mathbf{k}}$ act to preserve energy and enstrophy conservation under the replacement of $F_{\mathbf{k}}(\xi, h)$ by $F_{\mathbf{k}}(\chi, h)$. To see how this may be implemented, we form the equation of motion for $2Z_{\mathbf{k}}(t,t) = \{\langle \zeta_{\mathbf{k}}(t')\partial_t\zeta_{\mathbf{k}}(t) + \zeta_{\mathbf{k}}(t)\partial_{t'}\zeta_{\mathbf{k}}(t')\rangle\}_{t\to t'}$. From (3.4):

$$\partial_t Z_{\mathbf{k}}(t,t) = -2\int_0^t ds[\rho_{\mathbf{k}}(t,s)Z_{\mathbf{k}}(t,s) + \hat{\rho}_{\mathbf{k}}(t,s)X(t') + \langle \zeta_{\mathbf{k}}(t)F_{\mathbf{k}}(\chi,h,t)\rangle \tag{3.5}$$

and for $X_{\mathbf{k}}(t) \equiv \langle h_{\mathbf{k}}\zeta_{\mathbf{k}}(t)\rangle$

$$\partial_t X_{\mathbf{k}} = -\int_0^t ds[\rho_{\mathbf{k}}(t,s)X_{\mathbf{k}}(s) + \hat{\rho}(t,s)H_{\mathbf{k}}] + \langle h_{\mathbf{k}}F_{\mathbf{k}}(\chi,h)\rangle \tag{3.6}$$

In these equations $H_{\mathbf{k}} \equiv \langle h_{\mathbf{k}}h_{-\mathbf{k}}\rangle$. To evaluate $\langle\zeta_{\mathbf{k}}(t)F_{\mathbf{k}}(\chi,h,t)\rangle$, we note that from (3.4)

$$\left\{\begin{matrix}\zeta\\ h\end{matrix}\right\}(t) = \int_0^t ds\left\{\begin{matrix}G^{11}(t,s) & G^{12}(t,s)\\ G^{21}(t,s) & G^{22}(t,s)\end{matrix}\right\}\left\{\begin{matrix}F(\chi,h)\\ 0\end{matrix}\right\} \tag{3.7}$$

where $G^{ij}_{\mathbf{k}}(t,s)$ is:

$$\partial_t\left\{\begin{matrix}G^{11}(t,t') & G^{12}(t,t')\\ G^{21}(t,t') & G^{22}(t,t')\end{matrix}\right\} + \int_{t'}^t ds\left\{\begin{matrix}\rho(t,s) & \hat{\rho}(t,s)\\ 0 & 0\end{matrix}\right\}\left\{\begin{matrix}G^{11}(s,t') & G^{12}(s,t')\\ G^{21}(s,t') & G^{22}(s,t')\end{matrix}\right\}$$

$$= \left\{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\}\delta(t,t') \tag{3.8}$$

In (3.7) and (3.8), we have omitted the $\mathbf{k}$ subscripts for simplicity. Introducing (3.7) into (3.5) permits the evaluation of $\langle\zeta_{\mathbf{k}}F_{\mathbf{k}}(\chi,h)\rangle$ as:

$$\langle\zeta_{\mathbf{k}}F_{\mathbf{k}}(\chi,h)\rangle = \int_0^t dsG^{11}_{\mathbf{k}}(t,s)\langle F_{\mathbf{k}}(\chi,h,s)F_{\mathbf{k}}(\chi,h,t)\rangle \tag{3.9}$$

Notice that the right hand side ensemble average is, according to (3.2), and (3.4) a fourth moment of Gaussian variables, $(\xi_{\mathbf{k}},\ h_{\mathbf{k}})$, and hence becomes a bilinear function of $Z_{\mathbf{k}},\ X_{\mathbf{k}}$, and $H_{\mathbf{k}}$. The visco–elastic operators $\rho_{\mathbf{k}}(t,s),\ \hat{\rho}_{\mathbf{k}}(t,s)$ may now be determined by the constraint that for $\nu = 0$ the energy $((1/2)\sum_{\mathbf{k}} Z_{\mathbf{k}}/k^2)$ and enstrophy $(\sum_{\mathbf{k}}(Z_{\mathbf{k}}+2X_{\mathbf{k}}+H_{\mathbf{k}})$ are conserved. We shall not write these equations here. $\hat{\rho}(t,s)$ stems entirely from the interaction of topography with vorticity, and characterizes topographic drag on mode $\zeta_{\mathbf{k}}$. The realizability of $Z_{\mathbf{k}}(t,t') \equiv \langle\zeta_{\mathbf{k}}(t)\zeta_{-\mathbf{k}}(t')\rangle$ is fully guaranteed by (3.4). The full two–time form of $Z_{\mathbf{k}}(t,t')$ follows from (3.4) by multiplying (3.4) by $\zeta_{-\mathbf{k}}(t')$ and ensemble averaging.

We now simplify the notation by denoting the convolution $\int_0^t dsx(t,s)y(s,t')$ as $x * y$. Where useful, we denote the operator acting on $\mathbf{G}(t,t')$ in (3.8) as $\{\mathbf{G}\}^{-1}$. Then, (3.5) becomes more simply,

$$\{G^{11}_{\mathbf{k}}\}^{-1} * Z_{\mathbf{k}} = G^{11} * \langle FF^{\dagger}\rangle - \hat{\rho}_{\mathbf{k}} * X^{\dagger} \tag{3.10}$$

Where † denotes the transpose in (t,t'). Similarly, with the equation for $X(t)$

$$\{G^{12}_{\mathbf{k}}\}^{-1} * X_{\mathbf{k}} = -\hat{\rho} * H_{\mathbf{k}} \tag{3.11}$$

It is of interest to examine the stationary state form of these equations (assuming either inviscid flow, or an additional forcing, compatible with stationary turbulence). In that case, $X(t,t') = X(\tau \equiv t-t')$ and the equations for the Fourier transforms

$$\tilde{X}(\omega) \equiv \int_{-\infty}^{\infty} d\tau X(\tau) \tag{3.12}$$

are of the form:

$$i\omega \tilde{Z}_{\mathbf{k}}(\omega) = -\tilde{\rho}_{\mathbf{k}}(\omega)\tilde{X}_{\mathbf{k}}(\omega) - \tilde{\hat{\rho}}_{\mathbf{k}}(\omega)X_{\mathbf{k}}(\omega) + G^{11}_{\mathbf{k}}(\omega)\langle FF^{\dagger}\rangle(\omega) \tag{3.13}$$

and

$$0 = -[\tilde{\rho}_{\mathbf{k}}(\omega)X_{\mathbf{k}} + \tilde{\hat{\rho}}_{\mathbf{k}}(\omega)H_{\mathbf{k}}]\delta(\omega) + \langle h_{\mathbf{k}}F_{\mathbf{k}}(\chi,h) \tag{3.14}$$

Here we have used the fact that $\tilde{X}_{\mathbf{k}}$ and $\tilde{H}_{\mathbf{k}} \sim \delta(\omega)$, for stationary turbulence. It follows from (3.13) and (3.14) that $Z_{\mathbf{k}}$ must have a static component, $Z^{static}_{\mathbf{k}}(\tau)$ independent of τ:

$$\hat{\rho}_{\mathbf{k}}(0)Z^{\rm static}_{\mathbf{k}} = -\tilde{\hat{\rho}}_{\mathbf{k}}(0)H_{\mathbf{k}} \tag{3.15}$$

We recall that $\rho_{\mathbf{k}}(0)$ and $\hat{\rho}_{\mathbf{k}}(0)$ are $\int d\tau \exp(i\omega\tau)(\rho_{\mathbf{k}}(\tau),\ \hat{\rho}_{\mathbf{k}}(\tau)$ evaluated at $\omega = 0$.

For regions of scale–size for which dissipation plays little role, it may be shown that $\tilde{\rho}(0)_{\mathbf{k}} \simeq \tilde{\hat{\rho}}(0)_{\mathbf{k}}$, (Holloway, (1976), Herring (1975)) so that $Z_{\mathbf{k}} \sim= H_{\mathbf{k}}$, and $X_{\mathbf{k}} = -H_{\mathbf{k}}$. This result is plausible if we argue that if $\mathbf{u}$ is large (in some nominal sense) then $\partial_t\zeta$ in (1.19) is nominally small only if $\zeta = -h$, in detail. What the statistical theory gives is a way to quantify how strong this correlation is, and how it depends on the flow context. A close examination of (3.4), and the subsequent equations show that there is always the possibility of a static solution to the case of stationary turbulence, in the DIA, even for $h(\mathbf{x}) = 0$. By static solution, we mean one in which $\tilde{Z}_{\mathbf{k}}(\omega) \sim \delta(\omega)$. Of course, such would require a static forcing; one which has no time dependence, but which varies from ensemble to ensemble. Although these static solutions to DIA are possible, studies of two–dimensional turbulence, and also convection (Herring, (1969)), suggest that they are not stable, even at quite small Reynolds number. But solutions to the DIA show a tendency for $Z_{\mathbf{k}}(\tau)$ to have a much longer timescale than $G(\tau)$, especially at small scales, where viscosity is important. Such is at variance with the fluctuation dissipation relation (2.9). In practice this has been found to make little difference, except for the present problem.

We should mention that (3.10) and (3.11) are similar to Markovian equations proposed by Holloway (1976, 1986). Roughly speaking, to obtain the latter, we parameterize the (t,t') arguments as $\exp(-\eta(k) \mid t-t' \mid)$, and assume that the relaxation of tipple moments is much faster than that of $Z_{\mathbf{k}}(t,t)$. Holloway compared his Markovian theory with DNS and found quite satisfactory results. The topographic drag (as contained in (3.4)–(3.10)) was also generalized for realistic anisotropic flow (on a β–plane), and compared to other results.

4. Concluding Comments

This review has focused on methods which assume that the full distribution function for the fluid vorticity field is, in some sense, close to multivariate Gaussian. We note that the inviscid equipartitioned distribution is itself multivariate Gaussian (in vorticity), and perhaps it is this fact, together with the tentative agreement of its predictions with simulations and observations, that has encouraged its generalization to forced dissipative flows. But the inviscid solution fails at small scales, where a divergence of total enstrophy is encountered. The latter is attributable to the absence of dissipative effects. Langevin models are able to give reasonable small–scale spectra, but of course have unrealistic lagged covariances. The latter problem is less severe for the stochastic models of Sec. 3, which have self–consistently–determined lagged covariances. Yet all these methods are unable to incorporate structures; indeed, their replacement of actual non–linearities with Gaussianly composed force fields would suggest their inadequacies in this respect. Space–filling structures are not the problem, since they may not produce strong non–Gaussian signals. It is only if structures become isolated that measures of non Gaussianity become significant. Such is the case for decaying homogeneous two–dimensional turbulence (McWilliams (1985)), as well as homogeneous quasigeostrophic flow (McWilliams *et al.* (1994)). But the presence of random topography, random forcing, and Rossby waves that exists in real applications may well disrupt the formation of isolated vortices and lead to a chaotic flow more commensurate with stochastic modeling. At least this has been found to be so in the study of scalar mixing by Bartello and Holloway (1991).

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