# Conservation

## in numerical model dynamical cores

Tuesday 23 September, 2008

## Outline

- Conservation properties of the continuous adiabatic frictionless governing equations
- What conservation properties can we obtain in numerical models?
- Which conservation properties are most relevant/important?

  Finite resolution effects; the adiabatic frictionless **limit**

  Spurious sources vs physical sources

(Closely following T 2008, J. Comput. Phys.)

## Conservation properties of the continuous adiabatic frictionless governing equations

Flux form conservation laws

Lagrangian conservation laws

Conserved integral quantities

Kinematic identities

## Flux form conservation laws

$$\frac{\partial A}{\partial t} + \nabla.\mathbf{F} = 0,$$

| Quantity | $A$ | $\mathbf{F}$ |
|---|---|---|
| **Mass** | $\rho$ | $\rho\mathbf{u}$ |
| **Angular momentum** | $\rho\widehat{\mathbf{z}}.\left[\mathbf{r}\times(\mathbf{u}+\mathbf{\Omega}\times\mathbf{r})\right]$ | $\mathbf{u}A + p\widehat{\mathbf{z}}\times\mathbf{r}$ |
| **Energy** | $\rho\left(\frac{1}{2}\mathbf{u}^2 + c_v T + \Phi\right)$ | $\mathbf{u}\left(A+p\right)$ |

## Lagrangian conservation laws

$$\frac{D\chi}{Dt} = 0 \quad \Rightarrow \quad \frac{Df(\chi)}{Dt} = 0$$

**Potential temperature** $\theta$

**Potential vorticity** $Q = \boldsymbol{\zeta}.\nabla\theta/\rho$

**Specific tracer** $q$ or **tracer mixing ratio** $\eta$

Each Lagrangian conservation law generates an infinite family of flux form conservation laws

$$\frac{\partial}{\partial t}\left(\rho f(\chi)\right) + \nabla.\left(\rho \mathbf{u} f(\chi)\right) = 0$$

## Conserved integral quantities

| | |
|---|---|
| **Mass per unit $\theta$** in an isentropic layer | $\mathcal{F}(\theta) = \int \rho / \lvert\nabla\theta\rvert \, dA$ |
| **Mass per unit $\theta$** in an isentropic layer within a material contour | $\mathcal{M} = \int_D \rho / \lvert\nabla\theta\rvert \, dA$ |
| **Absolute circulation** around an isentropic material contour | $\mathcal{C} = \oint_\Gamma \mathbf{v}_a . d\mathbf{r} = \int_D \rho Q / \lvert\nabla\theta\rvert \, dA$ |

## Kinematic identities

The global integrals of horizontal divergence

$$\int_D \delta \, dA$$

and vertical component of vorticity

$$\int_D \zeta \, dA$$

must vanish on any isosurface of the vertical coordinate that wraps the sphere.

## Techniques for obtaining or approximating conservation properties in numerical models

## 1 Predict the desired variable using a discrete flux form conservation law

$$\frac{A_j^{n+1} - A_j^n}{\Delta t} + \frac{F_{j+1/2} - F_{j-1/2}}{\Delta x_j} = 0$$

E.g. $A$ might be $\rho$ times specific humidity.

## Techniques for obtaining or approximating conservation properties in numerical models

## 2 Impose discrete analogues of special cancellations

E.g. Coriolis terms on the C-grid; Arakawa Jacobian

In some cases there are systematic ways of deriving such schemes using ‘mimetic’ methods or Poisson bracket and Nambu bracket ideas

(May only work globally)

## Techniques for obtaining or approximating conservation properties in numerical models

## 3 Lagrangian conservation properties

Use a Lagrangian solution technique (or hybrid such as CASL, PIC)

Use Lagrangian or quasi-Lagrangian coordinates (or at least vertical coordinate)

Nonoscillatory advection schemes (perhaps combined with ‘reverse engineering’)

## Techniques for obtaining or approximating conservation properties in numerical models

## 4 Special forms of scale-selective dissipation

E.g. Anticipated potential vorticity method

E.g. Energy backscatter

**Which conservation properties are most relevant/important?**

**Which properties survive averaging? (Finite resolution effects)**

$$\frac{\partial \zeta}{\partial t} + \nabla.(\mathbf{v}\zeta) = 0$$

$$\frac{\partial \overline{\zeta}}{\partial t} + \nabla.(\overline{\overline{\mathbf{v}}\,\overline{\zeta}}) = \mathrm{SG}$$

SG term for vorticity is a divergence, but SG term for enstrophy is not.

**Which conservation properties are most relevant/important?**

**Adiabatic and frictionless, or adiabatic frictionless *limit* ?**

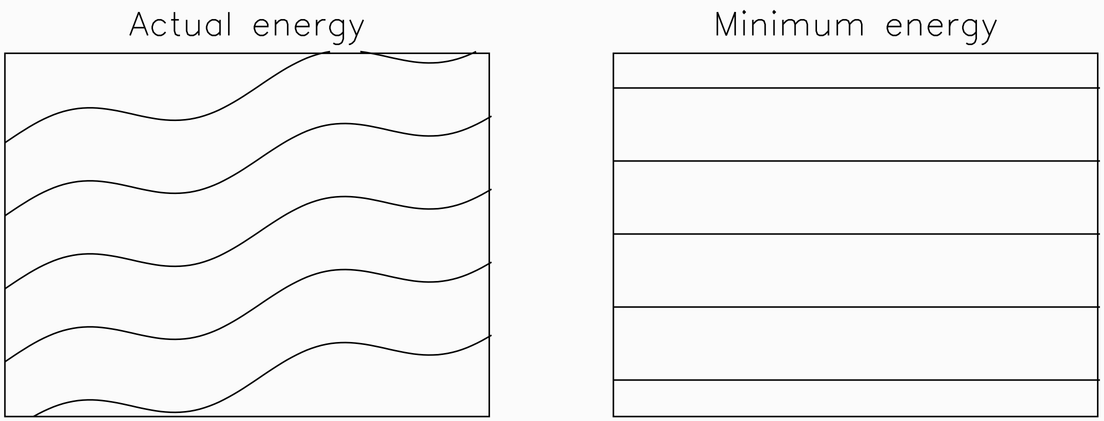
Quantities like tracer variance and potential enstrophy that cascade downscale are dissipated even in the limit of vanishing viscosity and thermal diffusivity. They are **non-Robust** invariants

**Dilemma**: attempt to conserve non-robust invariants, then dissipate them with a **sub-grid model**, or use inherently dissipative numerical methods **(ILES)**.

What about **energy**?

## Energy

Total energy ($\sim 3 \times 10^9 \mathrm{Jm}^{-2}$) is made up of available and unavailable contributions




| | Unavailable PE | Available PE | KE |
|---|---|---|---|
| Ratio: | 2000 | 4 | 1 |

Unavailable and available energy are separately conserved

Unavailable energy is a function of the $\mathcal{F}(\theta)$ - almost robust

Can conserve mass in each model isentropic layer by using $\theta$ as a vertical coordinate

There is some evidence that about 5-10% of available energy cascades downscale in the free atmosphere (the rest goes upscale before being dissipated by the boundary layer)

## Which conservation properties are most relevant/important?

## Spurious sources vs physical sources

Our numerical solutions should be accurate provided spurious numerical sources of conservable quantities are much weaker than true physical sources.

Conveniently expressed in terms of timescales.

## Summary of timescales

| Quantity | Robust | Cascade | Approx. timescale |
|---|---|---|---|
| Mass | Yes | | Infinite |
| Momentum | | | Minutes to hours |
| Angular momentum | | | 10 days (locally longer) |
| Potential enstrophy | | Yes | 10 days |
| Tracer variance | | Yes | 10 days |
| Unavailable energy | Almost | | 150 days |
| Available energy | | Yes (5-10%) | 20-30 days |
| Entropy | Almost | | Variable |

## Some questions to consider

- What can we conserve?
- What is most important to conserve?
- ILES approach, or conserve plus SG?
- Special measures for energy conservation?
- Lagrangian or quasi-Lagrangian vertical coordinate?
- What else is important, besides conservation?