# Magnetohydrodynamic Turbulence: challenges for numerical simulations

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Thanks to:
Fausto Cattaneo (U. Chicago),
Jean Carlos Perez (U. New Hampshire),
Joanne Mason (U. Exeter, UK),
Vladimir Zhdankin (UW-Madison),
Konstantinos Horaites (UW-Madison),
Qian Xia (UW-Madison),

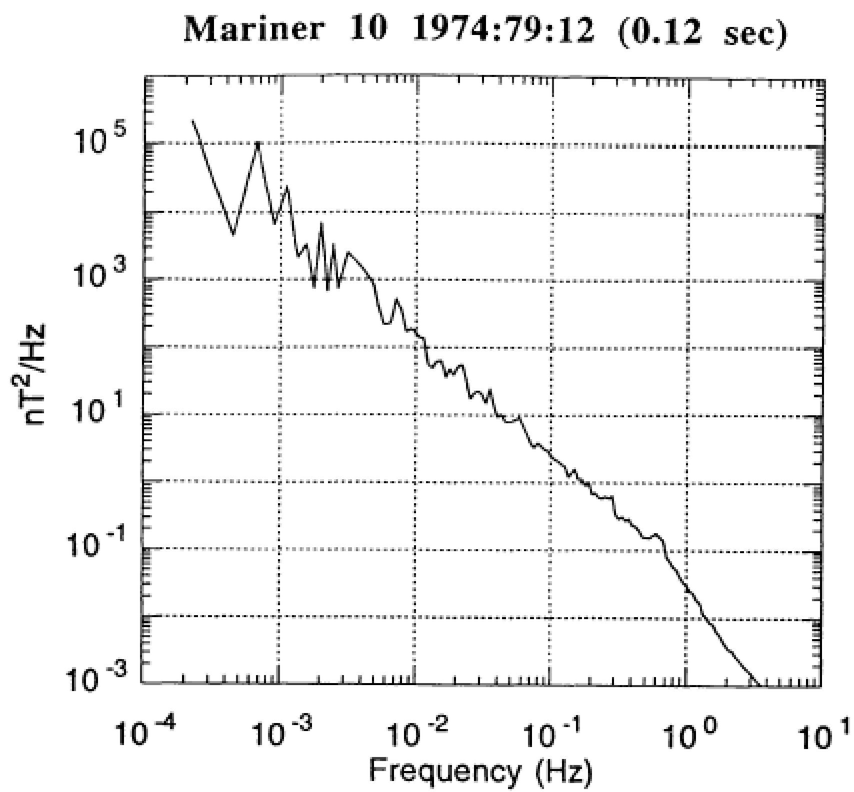
Two reviews: Tobias, et al in ``Ten Chapters on Turbulence” (2013);
Perez, et al   PRX 2 (2012) 041005

Center for Magnetic Self-Organization in Laboratory and Astrophysical Plasmas

Boulder, May 20, 2013

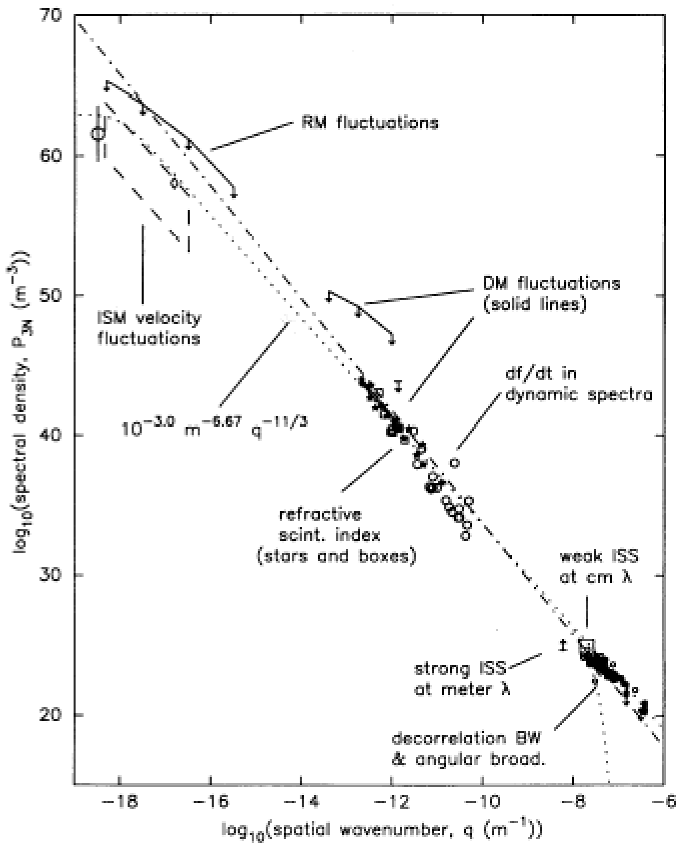
# Magnetic turbulence in nature

## energy spectra




Solar wind

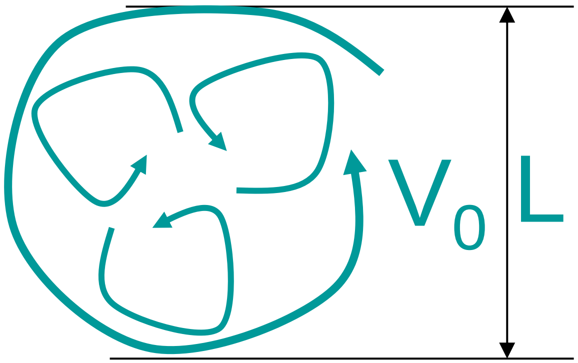
[Goldstein, Roberts, Matthaeus (1995)]

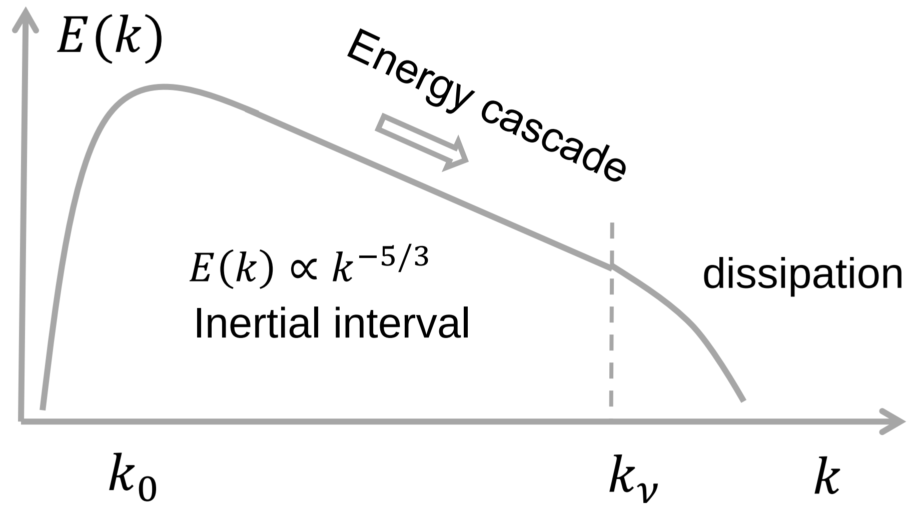



ISM

[Armstrong, Rickett, Spangler (1995)]

# Kolmogorov turbulence




$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p + \nu \nabla^2 v + f$$

$$Re = \frac{v_0 L}{\nu} \gg 1$$

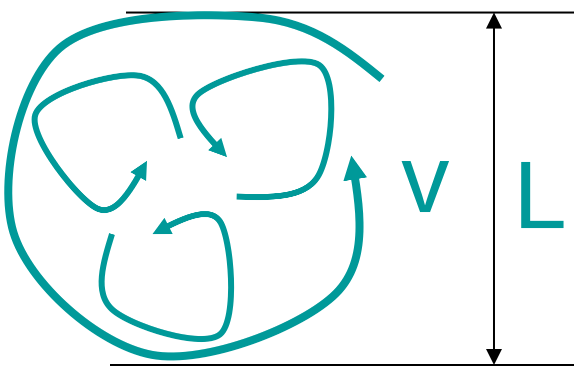
$$E(k) = C_k \epsilon^{2/3} k^{-5/3} f(k\eta)$$

$$f(0) = 1$$

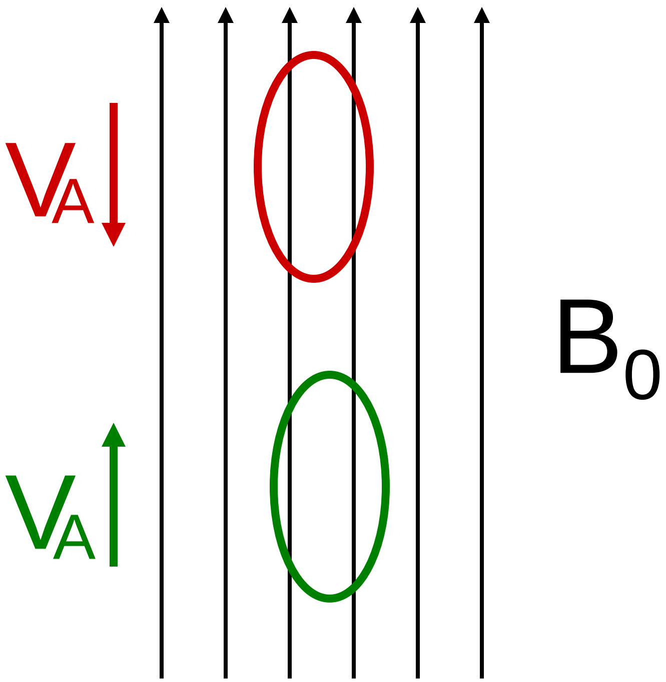
$$\eta = \nu^{3/4} \epsilon^{-1/4}$$

# MHD turbulence vs Hydrodynamic turbuence

HD turbulence:
interaction of eddies

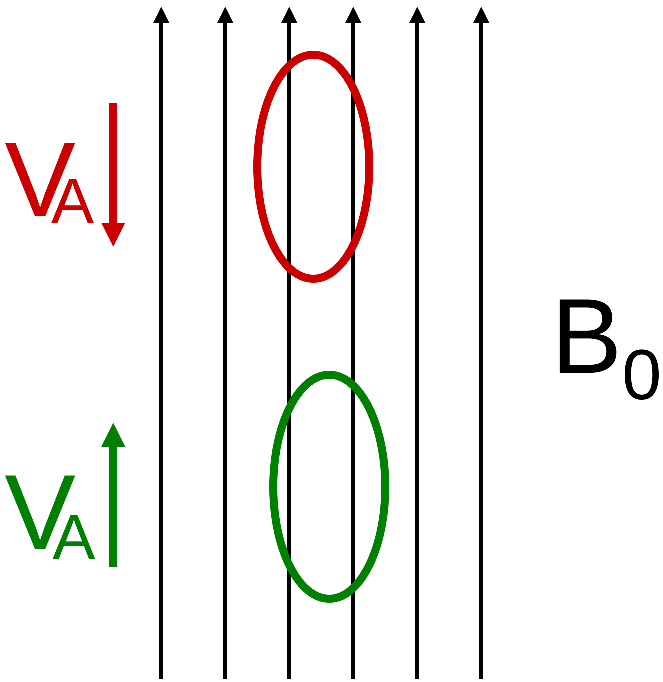



MHD turbulence:
interaction of wave packets
moving with Alfven velocities

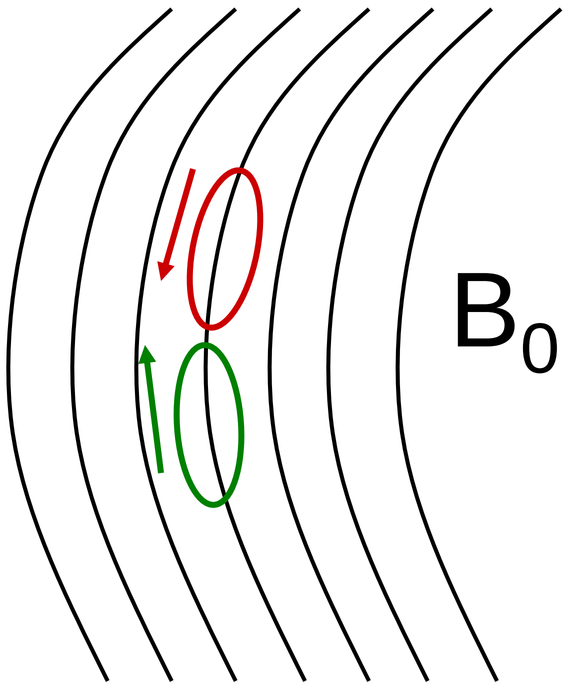



$$\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$$

# Guide field in MHD turbulence

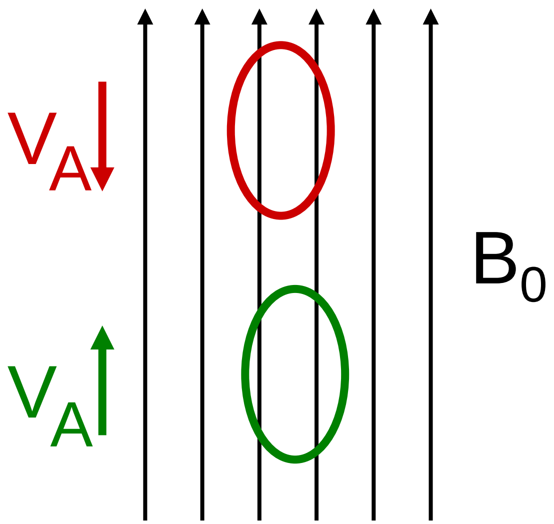


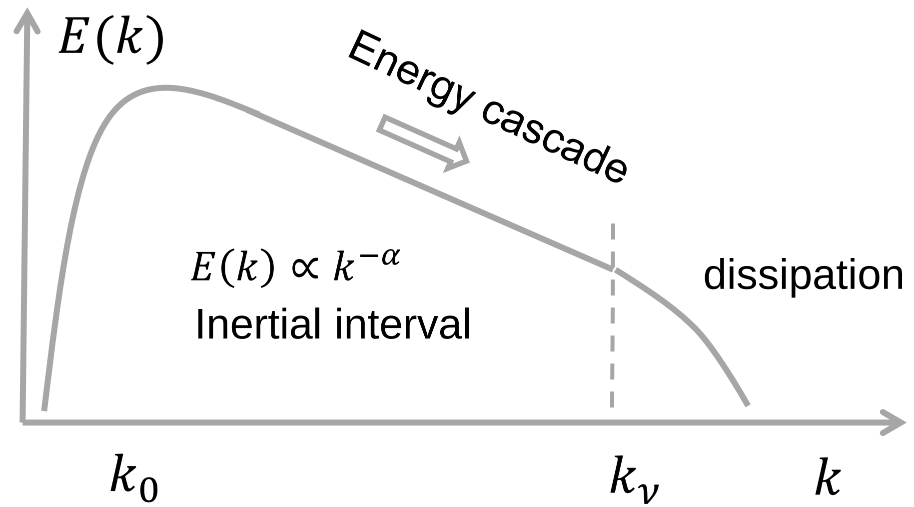

$B_0$ imposed by
external sources




$B_0$ created by
large-scale eddies

# Spectrum of MHD turbulence




Dimensional arguments do not work
– new dimensional parameter $v_A$

$$E(k_\perp) = C_k^M \epsilon^{2/3} k_\perp^{-5/3} g(k_\perp \eta, k_\perp \Lambda)$$

Need to study nonlinear interaction in detail in order to find the dependence on $k\Lambda$ ($\Lambda$ is the outer scale).

weak turbulence:

$$g(0, k\Lambda) \sim (k\Lambda)^{-1/3}, \qquad E(k) \propto k_\perp^{-2}$$

strong turbulence:

$$g(0, k\Lambda) \sim (k\Lambda)^{+1/6}, \qquad E(k) \propto k_\perp^{-3/2}$$

# Magnetohydrodynamic (MHD) equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$
$$\partial_t \mathbf{B} = \nabla\times[\mathbf{v}\times\mathbf{B}] + \eta\nabla^2\mathbf{B}$$

Separate the uniform magnetic field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Introduce the Elsasser variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{1}{\sqrt{4\pi\rho_0}}\mathbf{b}$

Then the equations take a symmetric form:

$$\partial_t \mathbf{z}^{+} - (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{+} + (\mathbf{z}^{-}\cdot\nabla)\mathbf{z}^{+} = -\nabla P$$
$$\partial_t \mathbf{z}^{-} + (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{-} + (\mathbf{z}^{+}\cdot\nabla)\mathbf{z}^{-} = -\nabla P$$

With the Alfven velocity $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$

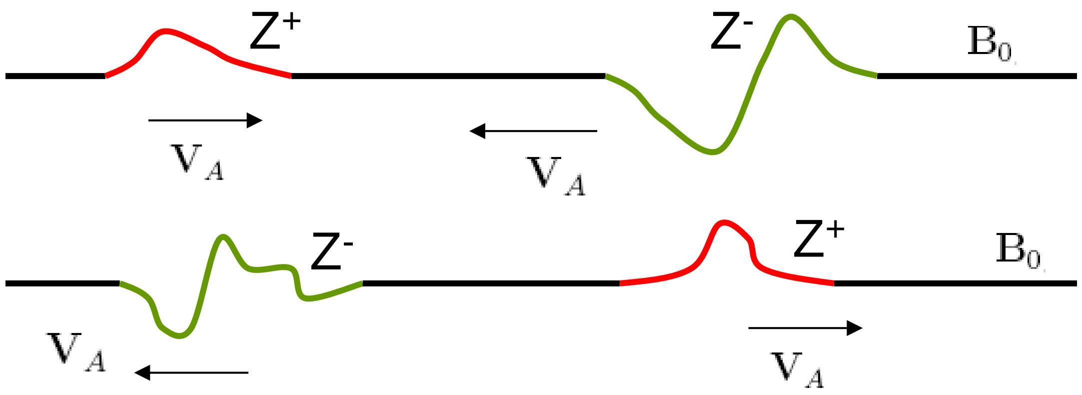
The uniform magnetic field mediates small-scale turbulence

# MHD turbulence: Alfvenic cascade

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

$$E^+ = \int (\mathbf{z}^+)^2 \, d^3x \qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$
$$E^- = \int (\mathbf{z}^-)^2 \, d^3x \qquad \equiv \qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$

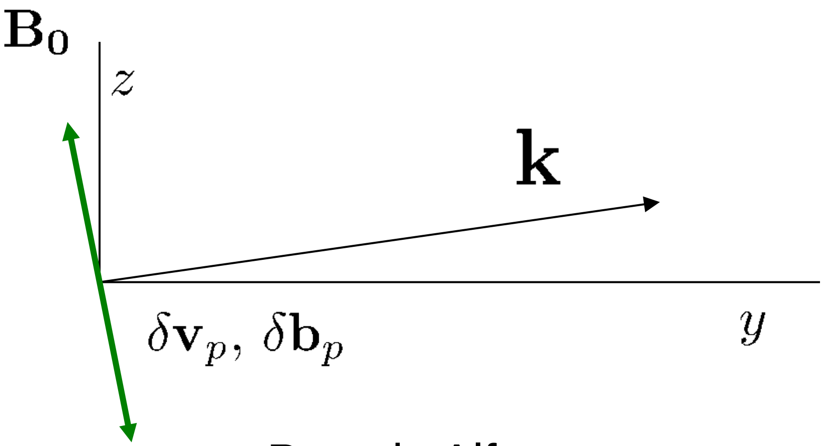



E$^+$ ~ E$^-$ : balanced case. E$^+$ $\gg$ E$^-$ : imbalanced case

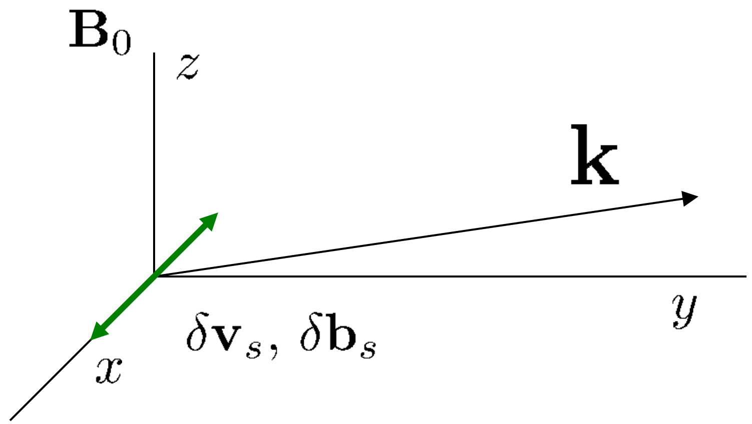
$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x = \frac{1}{4}(E^+ - E^-) \neq 0$$

# Anisotropy of MHD turbulence

Polarization of shear Alfven and pseudo Alfven waves




Pseudo Alfven




Shear Alfven

Cascade is dominated by shear Alfven modes, e.g.,

$$(z_s^{\pm}\cdot \nabla)z_p^{\mp} \gg (z_p^{\pm}\cdot \nabla)z_s^{\mp}$$

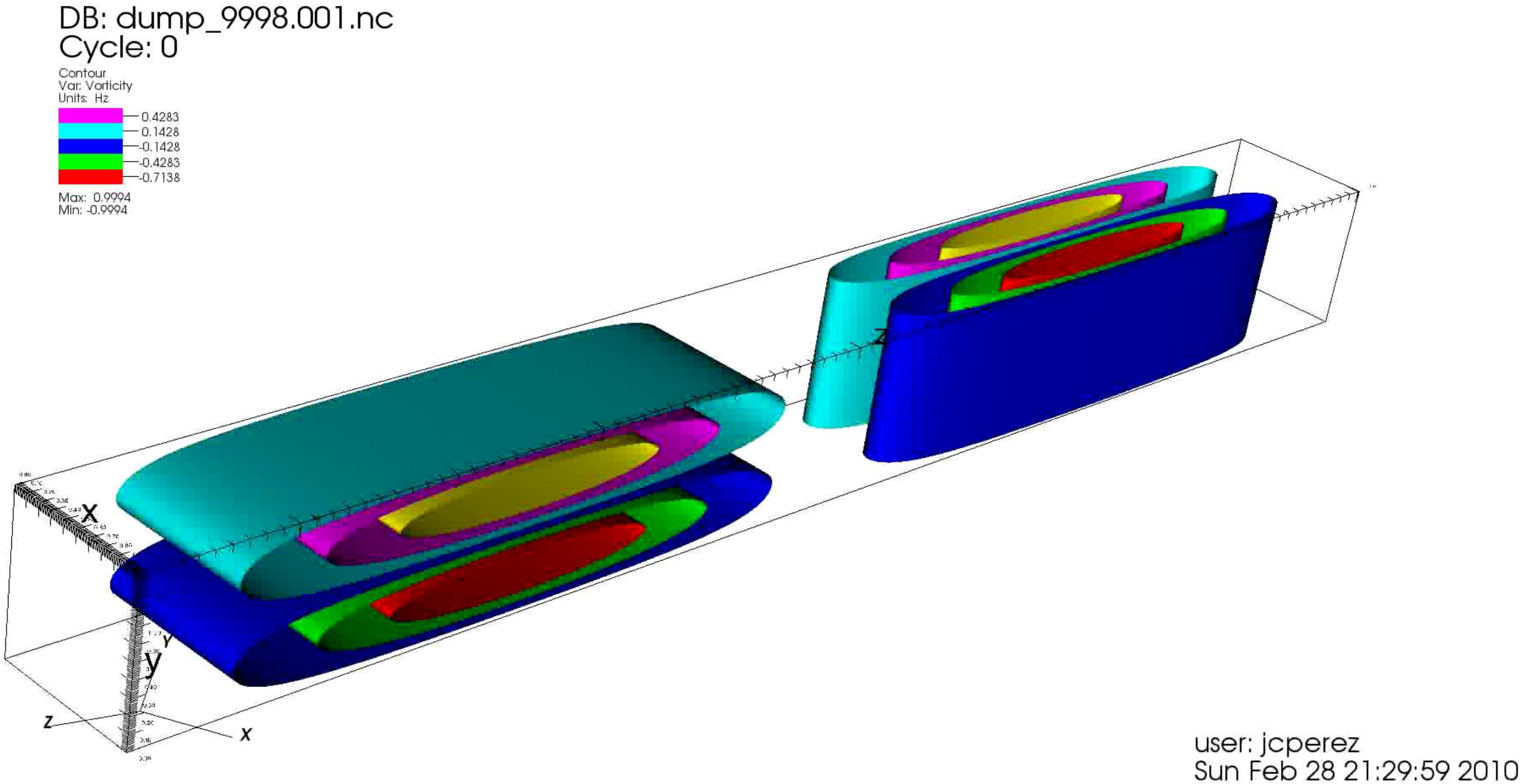
# Strength of interaction in MHD turbulence

$$\partial \mathbf{z}^{\pm} \mp \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{||} v_A) z^{\pm}} + \underbrace{(\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

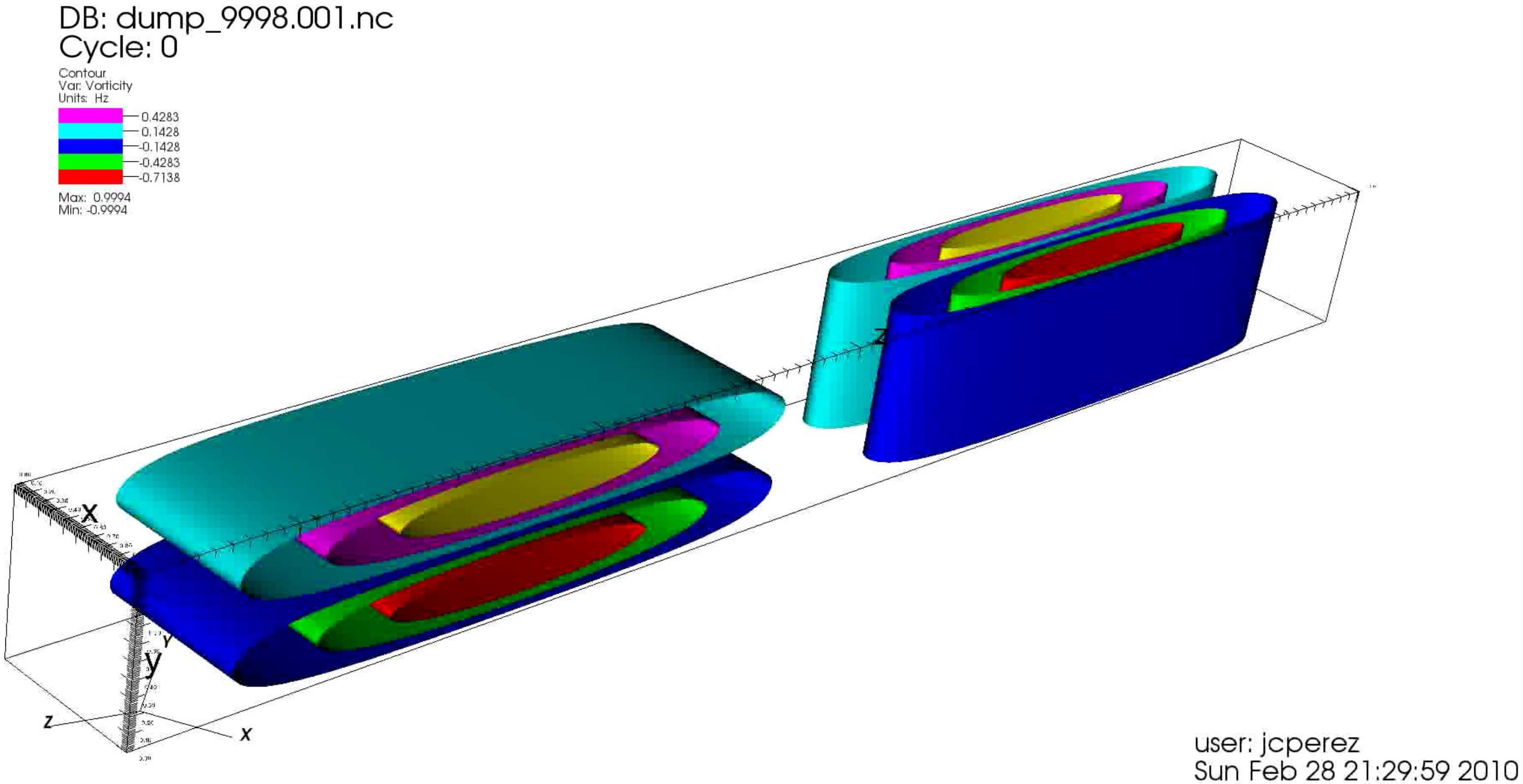
When $k_{||} v_A \gg k_{\perp} z^{\mp}$ turbulence is weak

When $k_{||} v_A \sim k_{\perp} z^{\mp}$ turbulence is strong

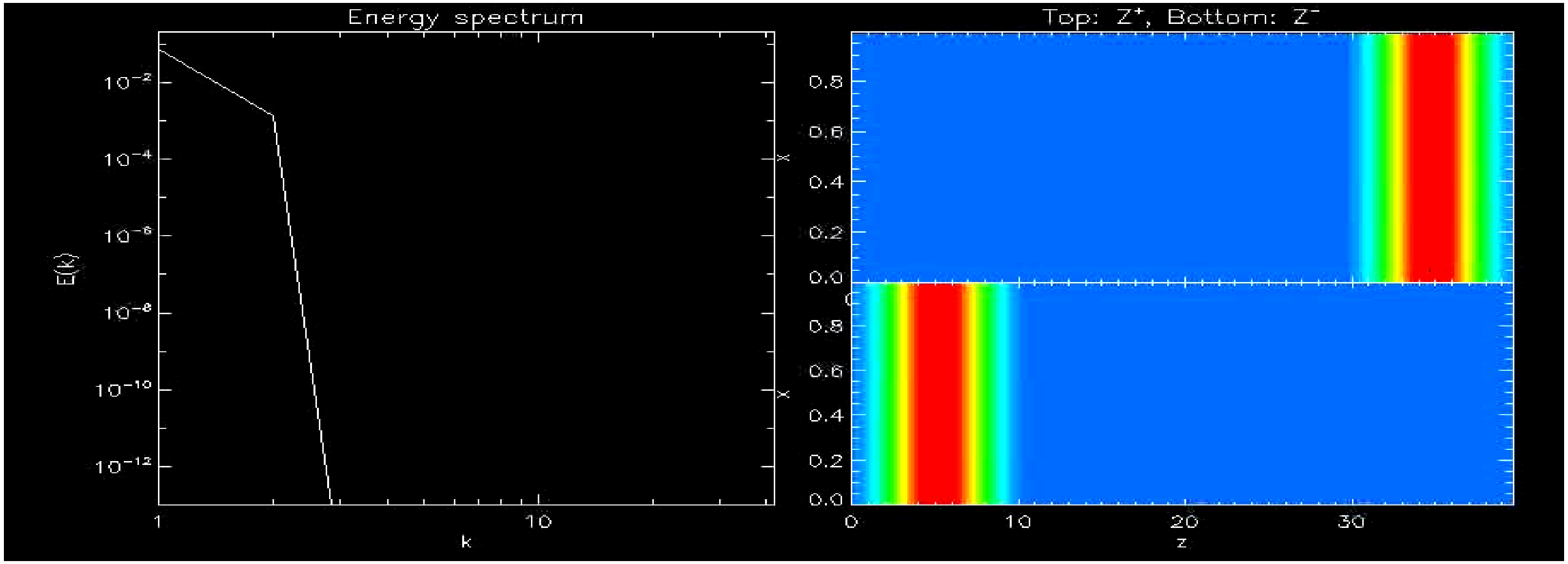
# MHD turbulence: collision of Alfven waves



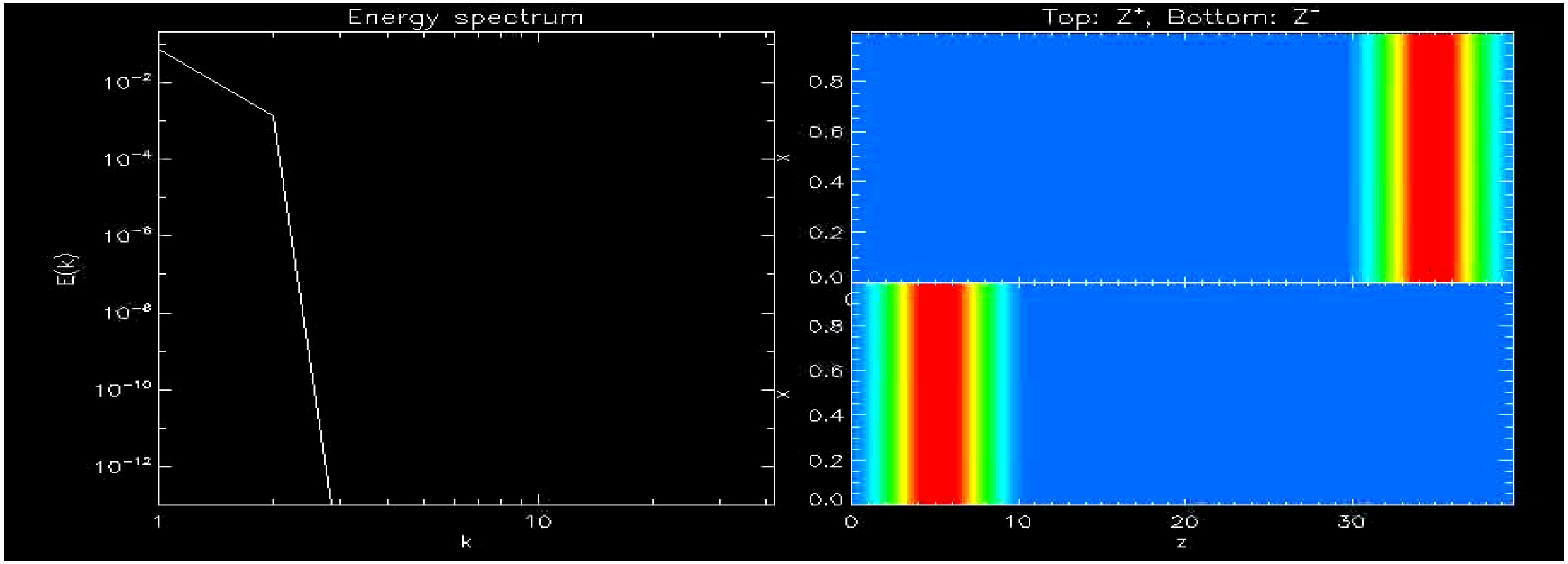
# MHD turbulence: collision of Alfven waves



# Strong MHD turbulence: collision of eddies

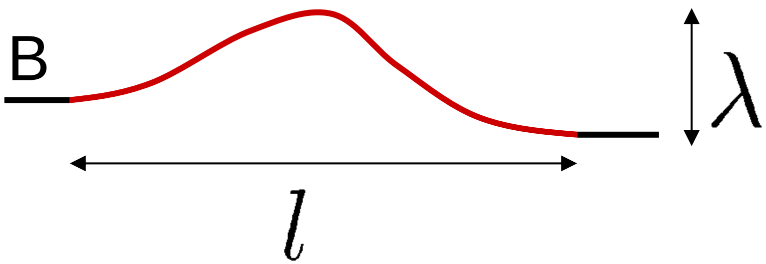


# Strong MHD turbulence: collision of eddies



# Goldreich-Sridhar spectrum of strong turbulence

## Anisotropy of “eddies”




$l \gg \lambda \qquad \delta v_\lambda \sim \delta b_\lambda \propto \lambda^{1/3}$

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$

$$\partial_t \mathbf{z}^- + \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^-}_{V_A/l \sim \delta b_\lambda/\lambda} = -\nabla P$$

Critical Balance

Energy spectrum:

$$\boxed{E(k_\perp) = C\epsilon^{2/3} k_\perp^{-5/3}}$$

no $\Lambda$ in the spectrum

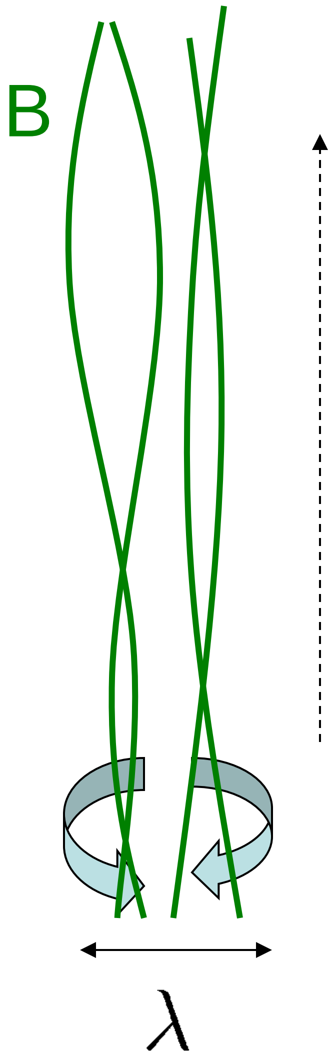
Anisotropy:

$$\boxed{(\lambda/\Lambda)^{2/3} \sim l/\Lambda}$$

should not be broken in numerical simulations!

[Goldreich & Sridhar 1995]

# Critical balance, anisotropy




$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$

$$\partial_t \mathbf{z}^- + \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)}_{V_A/l \sim \delta b_\lambda/\lambda}\mathbf{z}^- = -\nabla P$$

$$l \sim V_A \tau_\lambda \quad \Rightarrow \quad l \propto \lambda^{2/3}$$

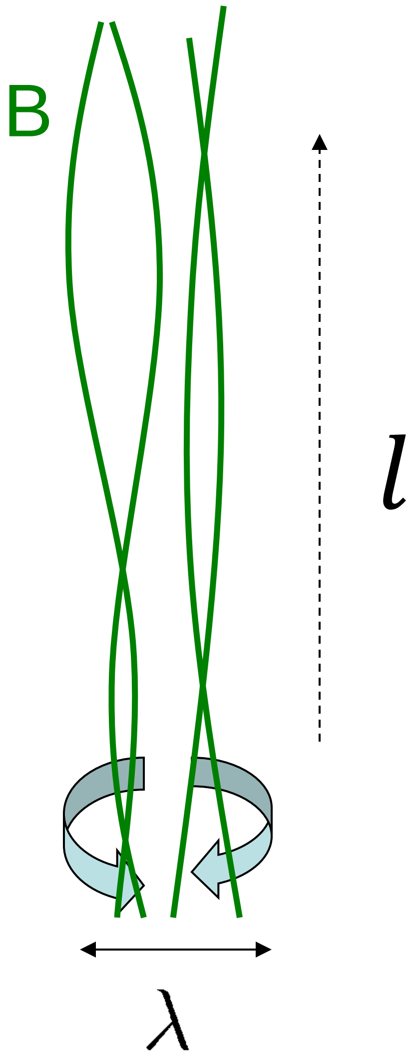
Causality — GS Critical balance

$$\tau_\lambda \sim \lambda/\delta v_\lambda$$

Correlation time of fluctuations, or eddy turnover time

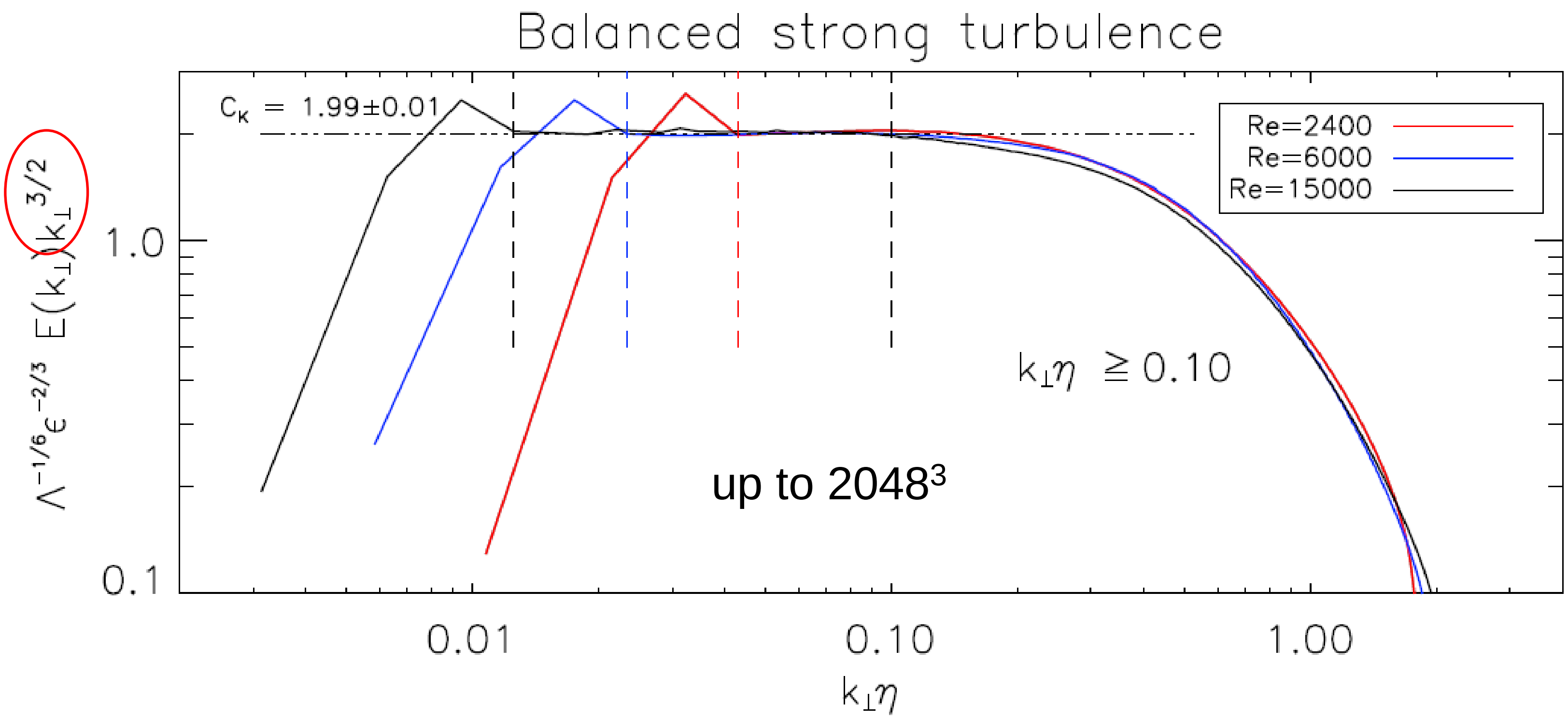
critical balance of strong turbulence is a consequence of causality

# Critical balance, anisotropy




Numerical scheme should correctly reproduce progressively increasing anisotropy of eddies at small scales

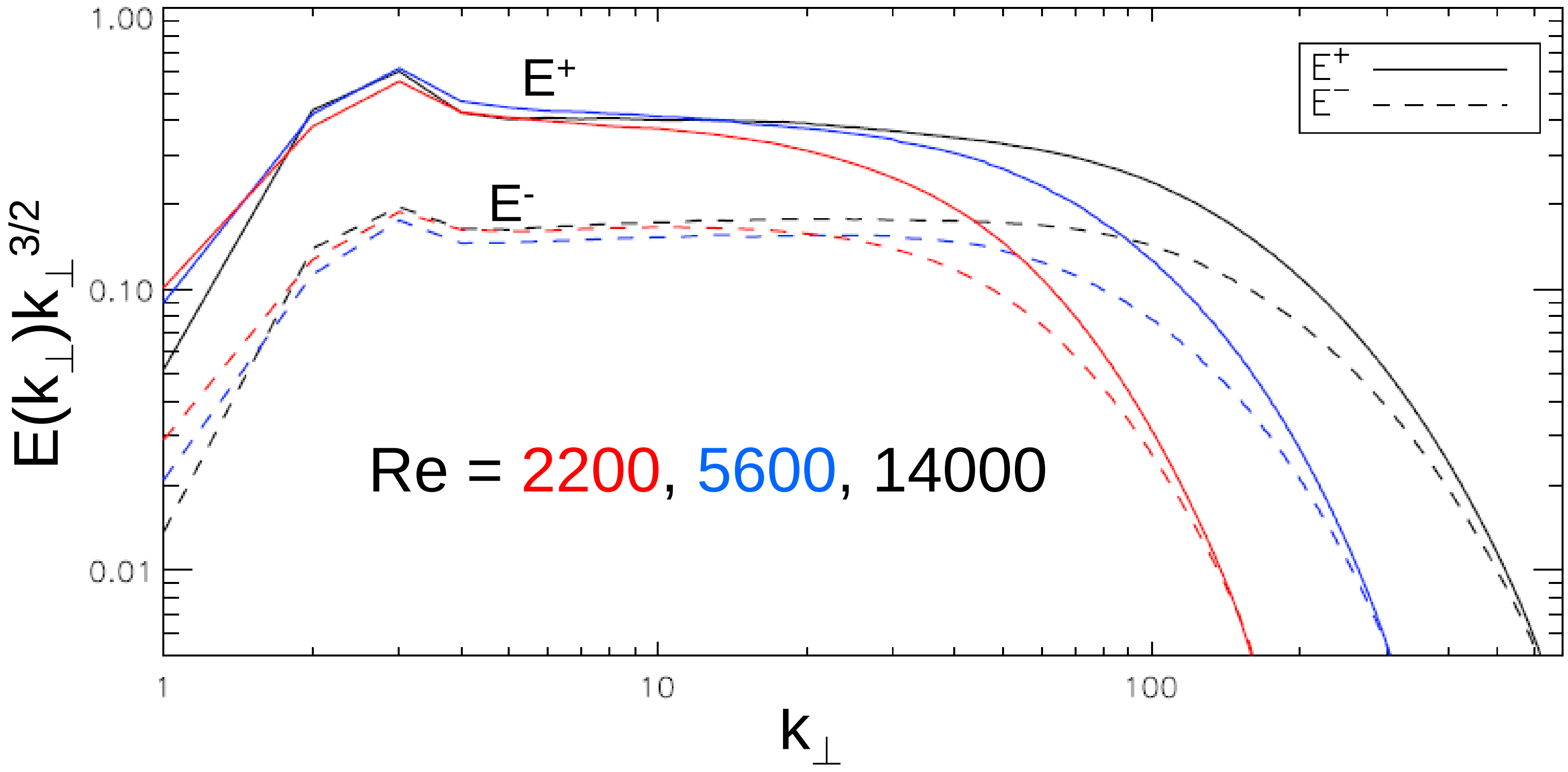
# Spectrum of strong MHD turbulence in DNS: balanced case




Computational resources: DoE 2010 INCITE,
Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

Perez et al, Phys Rev X (2012)

# Spectrum of strong MHD turbulence in DNS: imbalanced case



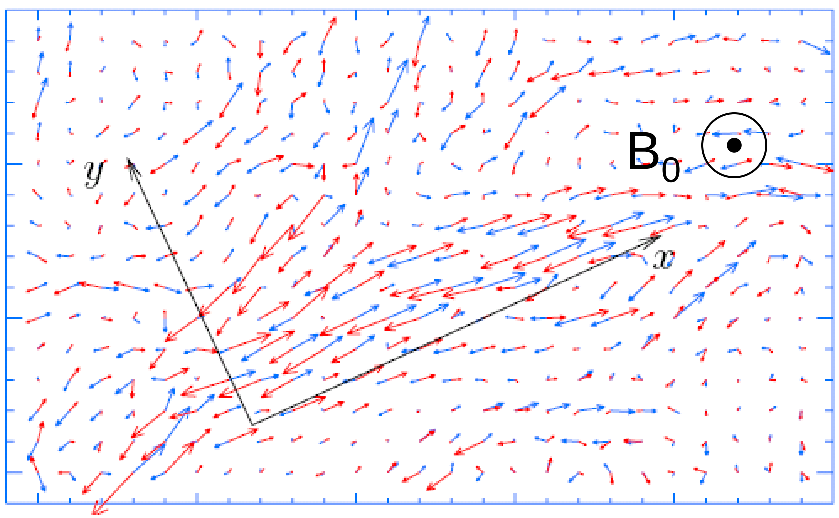

Computational resources: DoE 2010 INCITE,
Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

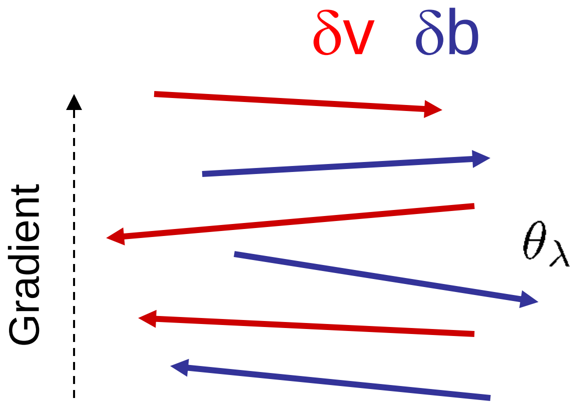
Perez et al, Phys Rev X (2012)

# Possible explanation of the -3/2 spectrum

## Dynamic Alignment theory

Fluctuations $\delta\mathbf{v}_\lambda$ and $\delta\mathbf{b}_\lambda$ become spontaneously aligned in the field-perpendicular plane within angle $\theta_\lambda$




$$(\mathbf{z}^{\pm}\cdot\nabla)\mathbf{z}^{\mp} \sim \theta_\lambda \delta v_\lambda^2/\lambda$$

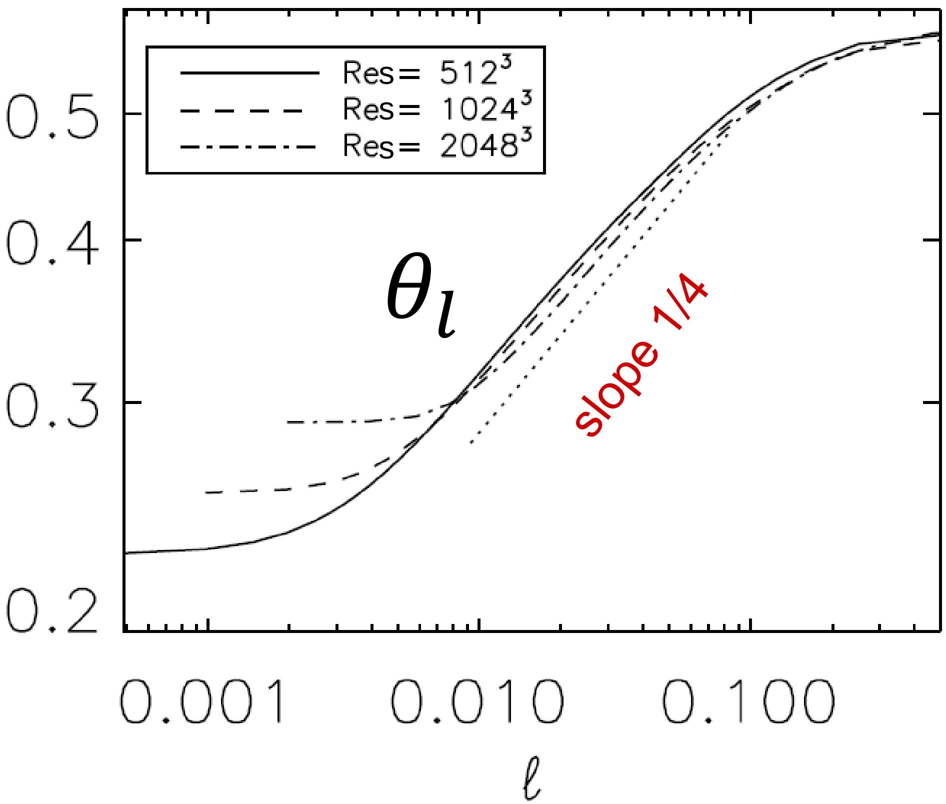
Nonlinear interaction is depleted

SB (2005, 2006)

# Numerical verification of dynamic alignment

$$S_{cross}(r) = \langle |\delta\tilde{\mathbf{v}}_r \times \delta\tilde{\mathbf{b}}_r|\rangle \qquad S_2(r) = \langle |\delta\tilde{\mathbf{v}}_r||\delta\tilde{\mathbf{b}}_r|\rangle$$

Alignment angle: $\theta_r \approx \sin(\theta_r) \equiv S_{cross}(r)/S_2(r)$




Magnetic and velocity fluctuations become progressively stronger aligned at smaller scales. Form sheet-like structures

$$\theta_l = (l/\Lambda)^{1/4}\theta_0$$

$$E(k_\perp) = C_k^M \epsilon^{2/3} k_\perp^{-5/3} (\mathrm{k}_\perp \Lambda)^{1/6}$$

$$l/\Lambda \sim (\lambda/\Lambda)^{1/2}$$

Mason et al 2011, Perez et al 2012

# Physics of the dynamic alignment

Hydrodynamics:

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v}$$

$$E = \tfrac{1}{2}\int \mathbf{v}^2(\mathbf{x})\, d^3x$$

MHD:

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$

$$\partial_t\mathbf{B} = \nabla\times[\mathbf{v}\times\mathbf{B}] + \eta\nabla^2\mathbf{B}$$

$$E = \tfrac{1}{2}\int(v^2+b^2)d^3x \qquad H^C = \int(\mathbf{v}\cdot\mathbf{b})d^3x$$

Energy E is dissipated faster than cross-helicity $H^C$

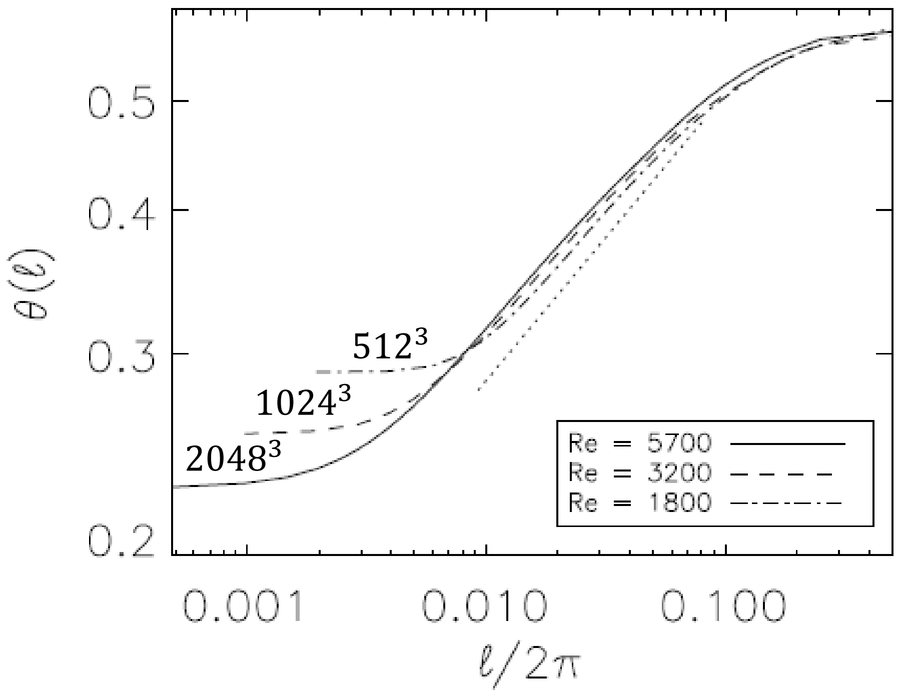
$$\frac{\delta}{\delta\mathbf{v}}\left[\int(v^2+b^2)d^3x - \lambda\int(\mathbf{v}\cdot\mathbf{b})\,d^3x\right] = 0$$

$$\frac{\delta}{\delta\mathbf{b}}\left[\int(v^2+b^2)d^3x - \lambda\int(\mathbf{v}\cdot\mathbf{b})\,d^3x\right] = 0$$

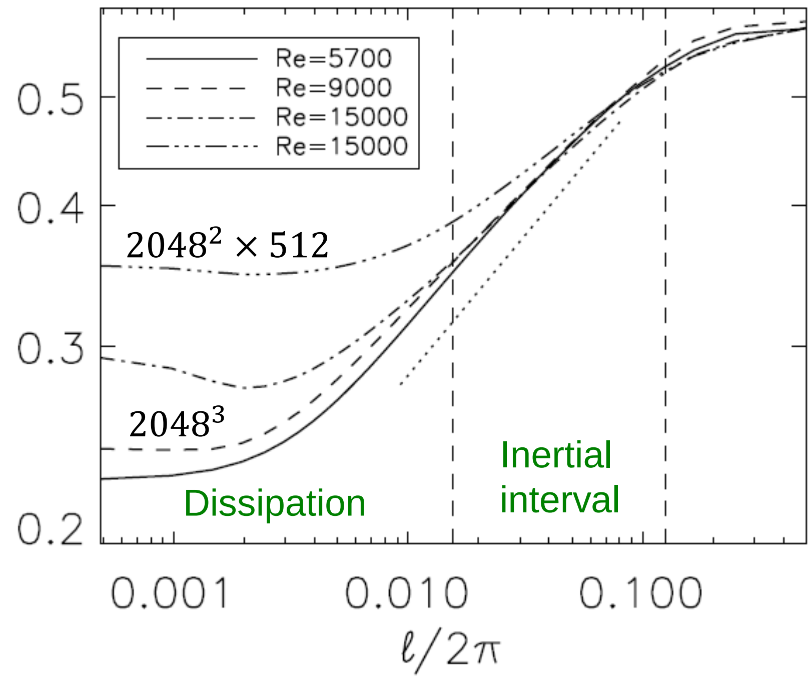
$$\Downarrow$$

$$\mathbf{v}(\mathbf{x}) = \pm\mathbf{b}(\mathbf{x})$$

# Angular alignment between $b_\lambda$ and $v_\lambda$
# role of resolution




Alignment is preserved in well resolved simulations at all scales, down to discretization scale




Alignment is broken in under-resolved simulations

Modeling of small scales should preserve the alignment

# Scale-Dependent Dynamic Alignment

3D anisotropic eddies

Goldreich-Sridhar 1995 “eddy”:

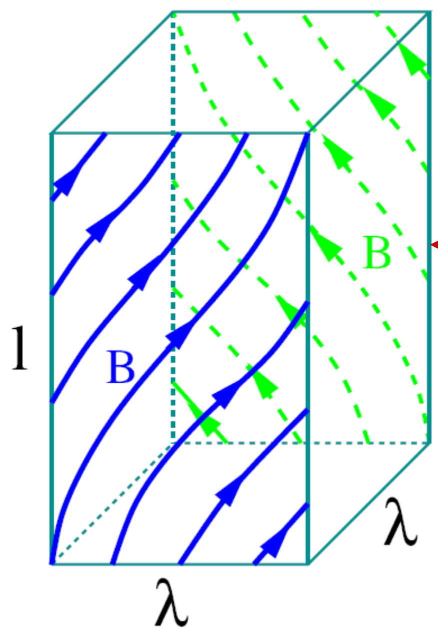
$\delta v_\lambda \propto \lambda^{1/3} \qquad l \sim \lambda^{2/3}$

line displacement: $\xi \propto \delta b_\lambda l \sim \lambda$

Dynamic alignment:

$\delta v_\lambda \propto \lambda^{1/4} \qquad l \sim \lambda^{1/2}$

line displacement: $\xi \propto \delta b_\lambda l \sim \lambda^{3/4}$

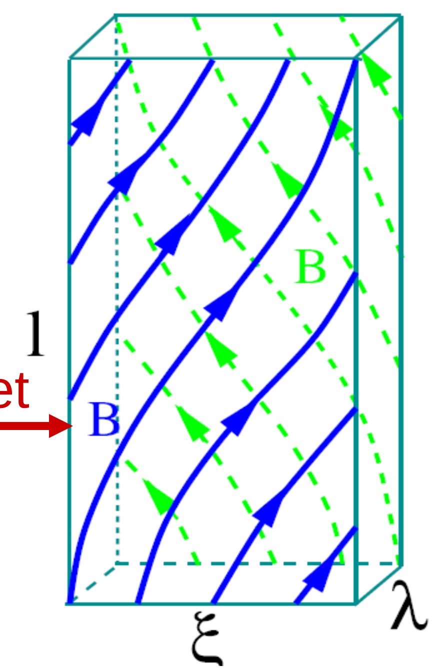



As the scale decreases,
λ→0,

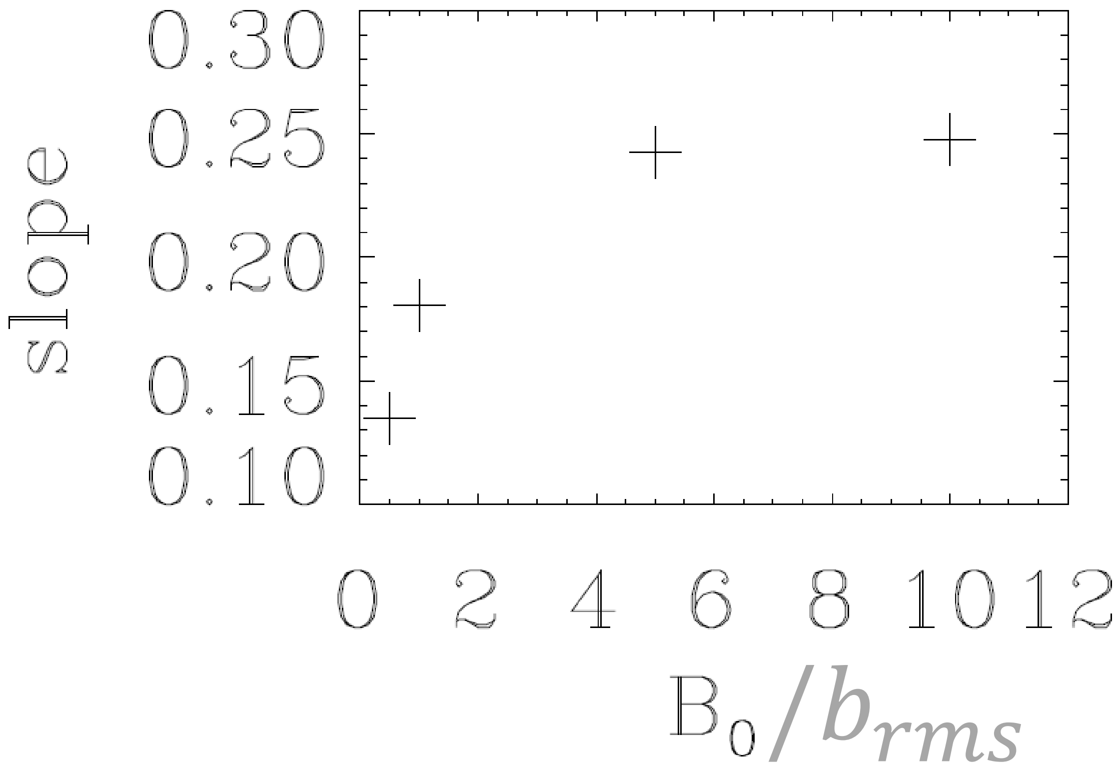
turns into filament

turns into current sheet

agrees with numerics!



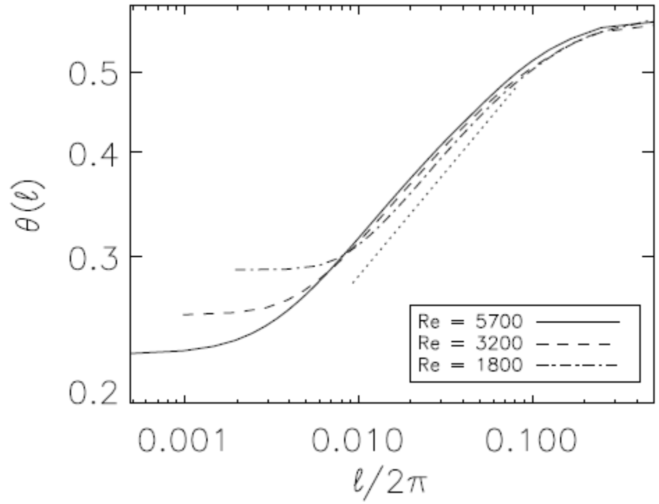
# How strong $B_0$ is "strong" ?

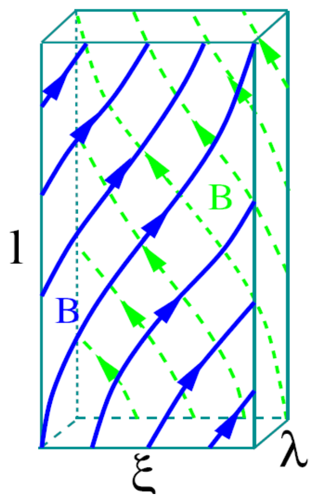



Large-scale magnetic field becomes essential when $B_0/b > 3$.

Mason et al 2006

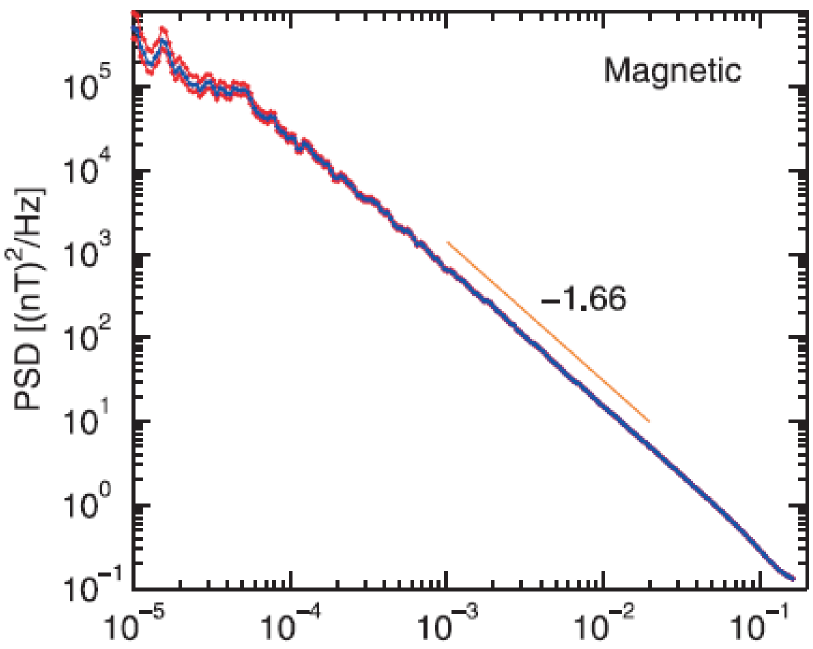
# Dynamic alignment and 3D anisotropy



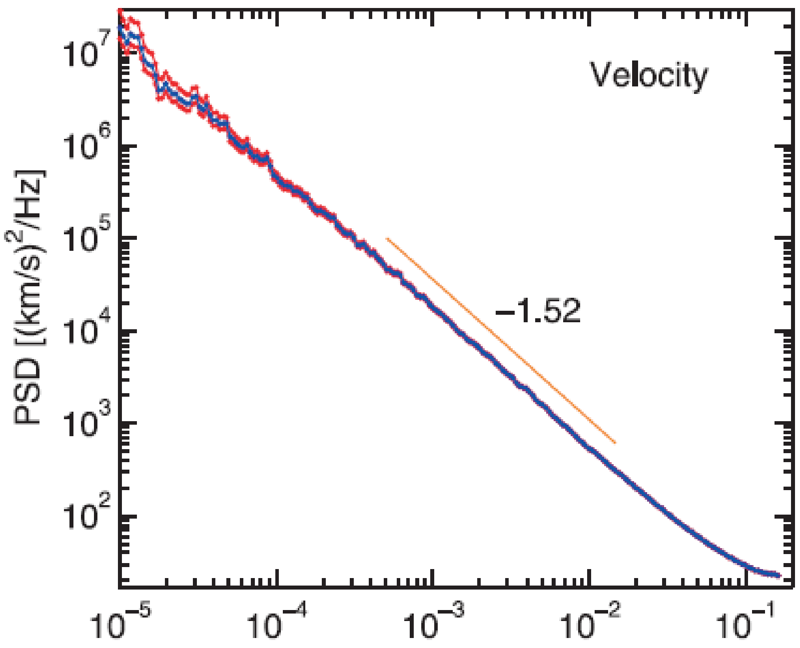




Numerical scheme should correctly reproduce progressively increasing alignment and 3D anisotropy of small scale eddies.

# Magnetic and kinetic spectra in the solar wind




frequency

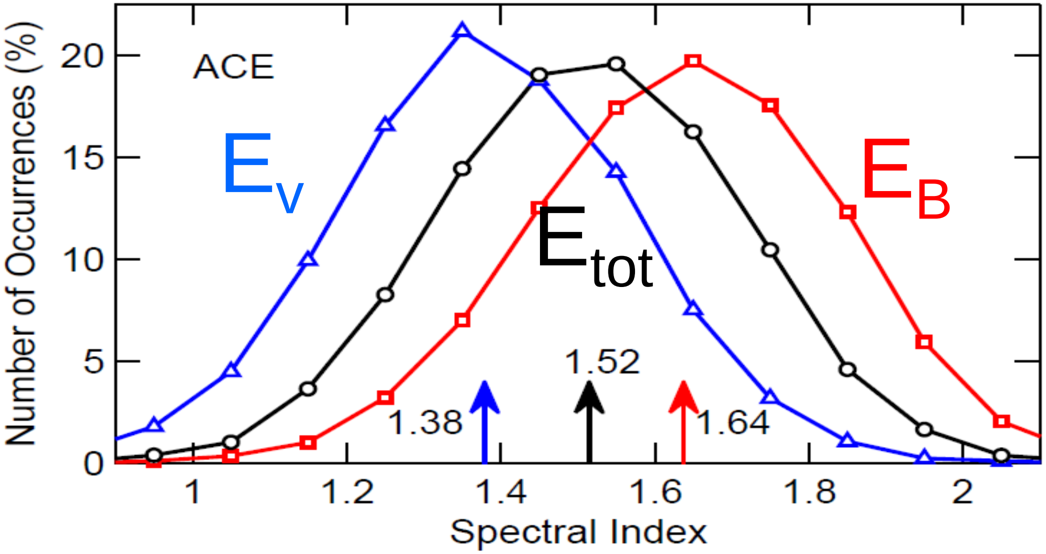



frequency

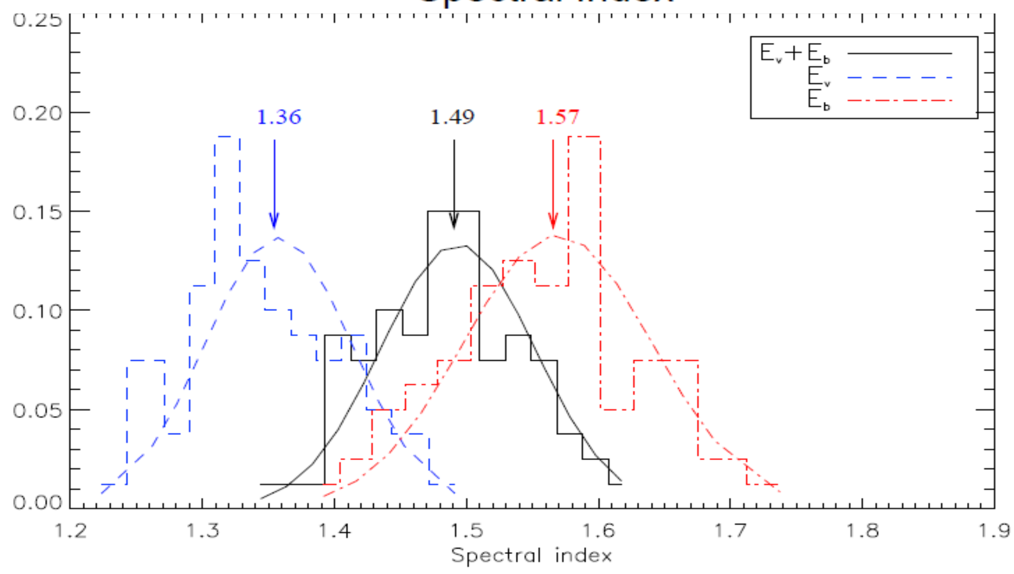
Podesta et al (2007)

# Energy spectra in the solar wind and in numerical simulations

Solar wind observations: spectral indices in 15,472 independent measurements. (From 1998 to 2008, fit from 1.8 $\times$ $10^{-4}$ to 3.9 $\times$ $10^{-3}$ Hz)




Numerical simulations: spectral indices in 80 independent snapshots, separated by a turnover time.




S.B., J. Perez, J Borovsky & J. Podesta (2011)

# Residual energy in MHD turbulence

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = q^r(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}') \qquad \neq 0$$

$$\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle$$ residual energy

-- need 3 correlation functions to describe two fields

-- residual energy plays important role in MHD dynamics.
e.g., the spectrum of weak MHD turbulence cannot be derived correctly without residual energy

# Residual energy in MHD turbulence

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$

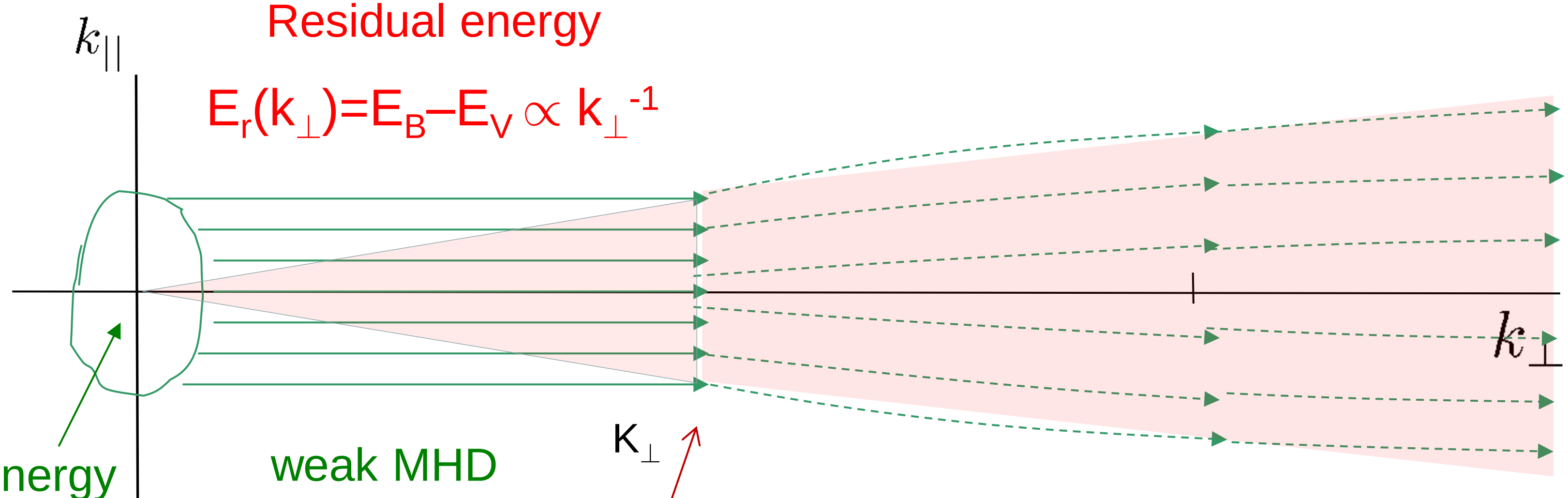
$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = q^r(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$

$$\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle$$ residual energy

Numerical simulations should
correctly reproduce small-scale residual energy

# Unifying picture of MHD turbulence




Residual energy

$E_r(k_\perp)=E_B-E_V \propto k_\perp^{-1}$

energy supply

weak MHD turbulence

$E_{tot}(k_\perp) \propto k_\perp^{-2}$

strong MHD turbulence

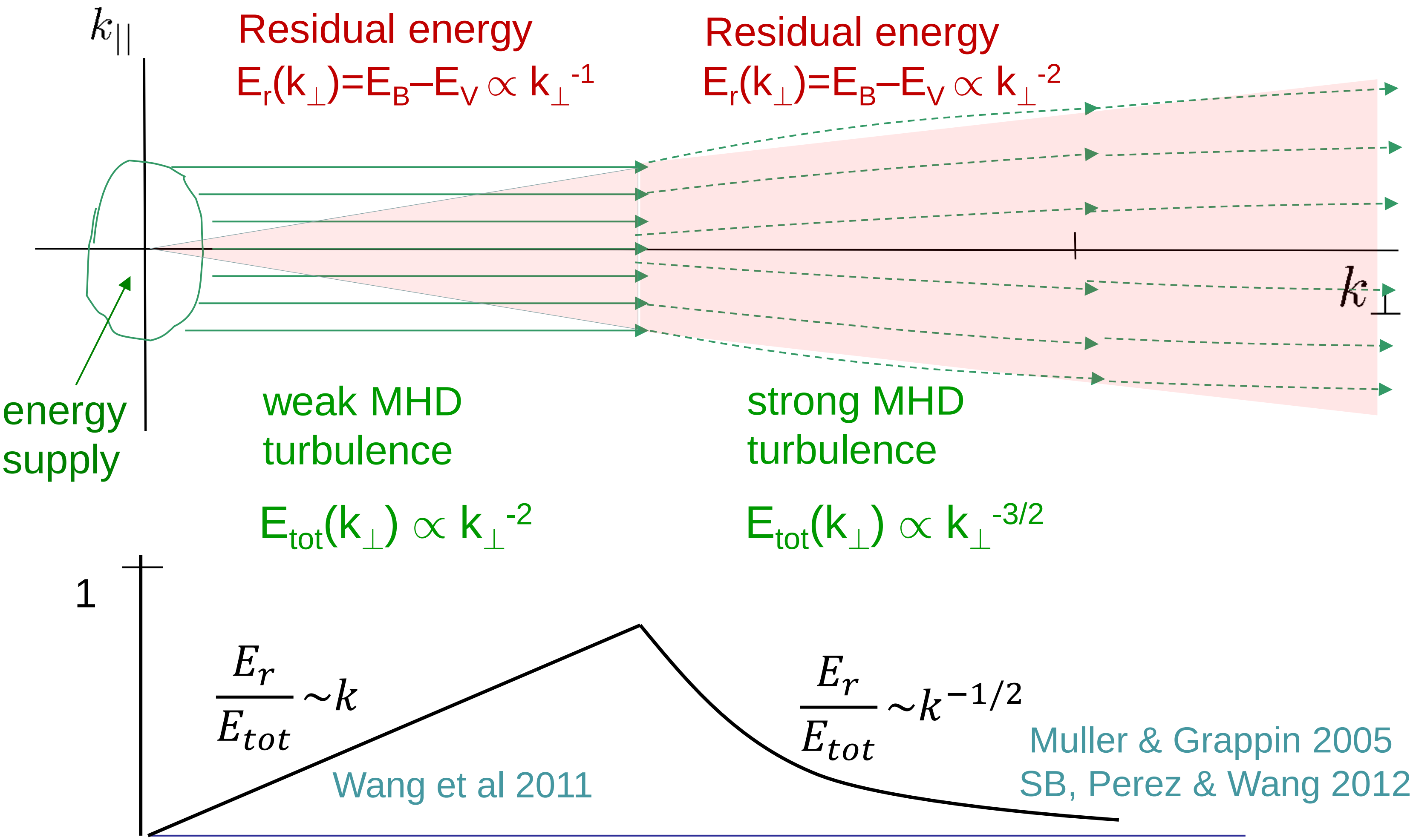
$E(k_\perp) \propto k_\perp^{-3/2}$

Here $E_r \sim E_{tot}$ !

Also, turbulence becomes strong!

Wang et al 2011

# Unifying picture of MHD turbulence



# Conclusions

- Numerical schemes should correctly capture the essential properties of MHD turbulence at small scales:

- $\frac{\delta b}{B_0} \leq 1/3$, otherwise turbulence is close to Kolmogorov;

- Scale dependent anisotropy of small-scale fluctuations with respect to the local magnetic field;

- Scale dependent correlation (alignment) of v and b fluctuations;

- residual energy.