Alfven waves: small-scale turbulence and large-scale structure

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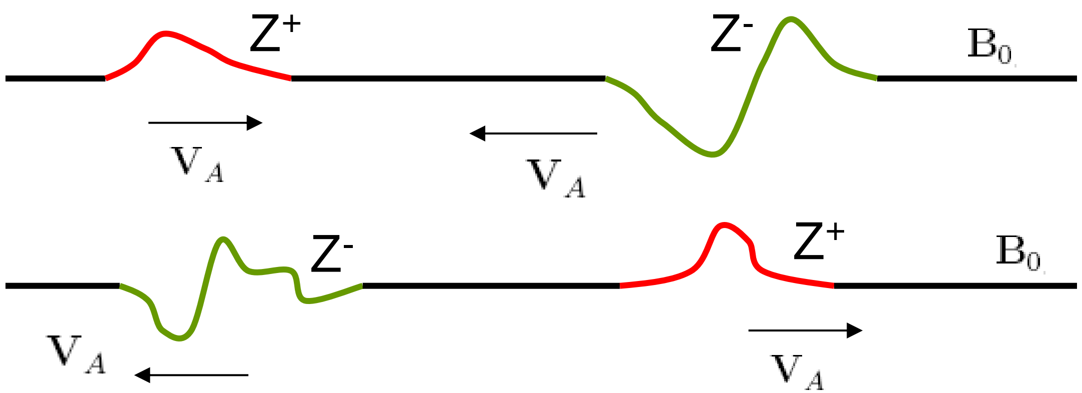
MHD turbulence: Alfvenic cascade

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

$$E^{+} = \int (\mathbf{z}^{+})^2 \, d^3x \qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$
$$E^{-} = \int (\mathbf{z}^{-})^2 \, d^3x \qquad \equiv \qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$



$E^+ \sim E^-$: balanced case. $E^+ \gg E^-$: imbalanced case

$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x = \frac{1}{4}(E^+ - E^-) \neq 0$$

Strength of interaction in MHD turbulence

$$\partial \mathbf{z}^{\pm} \mp \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{||} v_A) z^{\pm}} + \underbrace{(\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

When $k_{||} v_A \gg k_{\perp} z^{\mp}$ turbulence is weak

When $k_{||} v_A \sim k_{\perp} z^{\mp}$ turbulence is strong

Wave MHD turbulence: Phenomenology

Three-wave interaction of shear-Alfven waves

$$\omega(k) = |k_z| v_A$$

$$\begin{cases} \omega(k) = \omega(k_1) + \omega(k_2) \\ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \end{cases}$$

Only counter-propagating waves interact, therefore, k_{1z} and k_{2z} should have opposite signs.

⇓

Either $k_{1z} = 0$ or $k_{2z} = 0$

Wave interactions change $k_\perp$ but not k_z

At large $k_\perp$: $E(k_z, k_\perp) \propto g(k_z) k_\perp^{-\beta}$

Montgomery & Turner 1981, Shebalin et al 1983

Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

In the zeroth approximation, waves are not interacting. and z^+ and z^- are independent:

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = 0$$

When the interaction is switched on, the energies slowly change with time: $e^{\pm}(k_z, k_\perp, t)$

$$\partial_t \mathbf{z}^\pm - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^\pm + (\mathbf{z}^\mp \cdot \nabla)\mathbf{z}^\pm = -\nabla P$$

$$\partial_t \langle z^+ z^+ \rangle = \ldots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \ldots$$

$$\partial_t \langle z^- z^+ z^+ \rangle = \ldots \underbrace{\langle z^+ z^- z^+ z^+ \rangle} + \underbrace{\langle z^- z^- z^+ z^+ \rangle} + \underbrace{\langle z^- z^+ z^- z^+ \rangle} \ldots$$

split into pair-wise correlators using Gaussian rule

Weak turbulence: Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t \langle z^+ z^+ \rangle = \ldots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \ldots$$

$$\partial_t \langle z^- z^+ z^+ \rangle = \ldots \langle \cancel{z^+ z^- z^+ z^+} \rangle + \langle \underbrace{z^- z^- z^+ z^+} \rangle + \langle \underbrace{z^- z^+ z^- z^+} \rangle \ldots$$

split into pair-wise correlators using Gaussian rule

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

$$M_{k,pq} = \frac{\pi}{v_A} \frac{(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2 (\mathbf{k}_\perp \cdot \mathbf{p}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}$$

This kinetic equation has all the properties discussed in the phenomenology: it is scale invariant, z^+ interacts only with z^-, k_z does not change during interactions.

Weak turbulence: Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp)\right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

Statistically balanced case: $e^+ = e^-$

$$e^+(k_z, k_\perp) = e^-(k_z, k_\perp) = g(k_z) k_\perp^{-3}$$

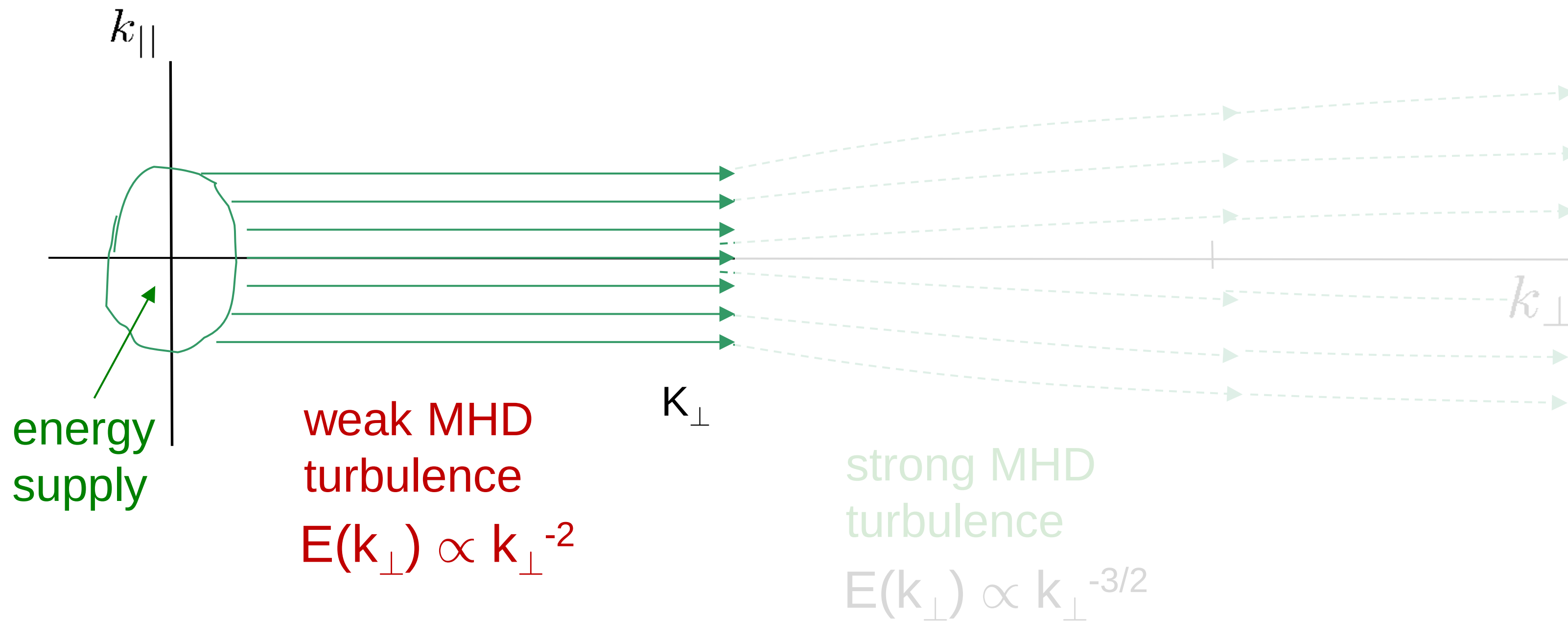
where $g(k_z)$ is an arbitrary function.

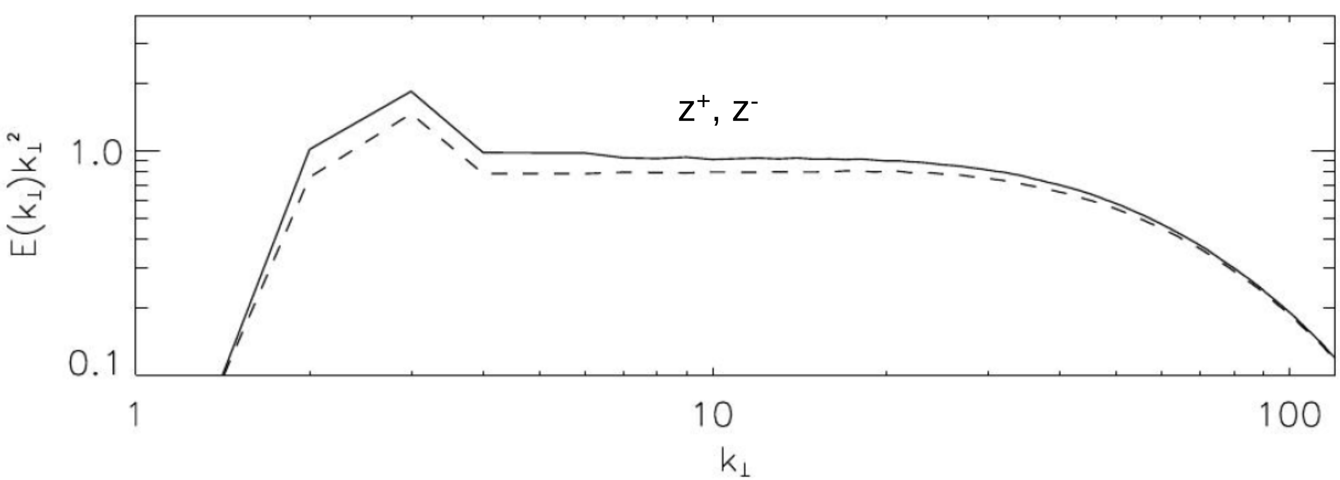
The spectrum of weak balanced MHD turbulence is therefore:

$$E^{\pm}(k_z, k_\perp) = e^{\pm}(k_z, k_\perp) 2\pi k_\perp \propto k_\perp^{-2}$$

Ng & Bhattacharjee 1996, Goldreich & Sridhar 1997

Weak MHD turbulence





SB & J. C. Perez (2009)

Imbalanced weak MHD turbulence

(where problems begin)

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp)\right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

The kinetic equation has a one-parameter family of solutions:

$$e^{+}(k_z, k_\perp) = g^{+}(k_z) k_\perp^{-3-\alpha}$$
$$e^{-}(k_z, k_\perp) = g^{-}(k_z) k_\perp^{-3+\alpha}$$

with $-1 < \alpha < 1$

What do these solutions mean? Hint: calculate energy fluxes.

Assume that e+ has the steeper spectrum and denote the energy fluxes ϵ^{+} and ϵ^{-}. Then $\epsilon^{+} > \epsilon^{-}$

and: $\alpha = f(\epsilon^{+}/\epsilon^{-})$

[Galtier et al 2000]

Imbalanced weak MHD turbulence

(where problems begin)

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp)\right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p\, d^2q$$

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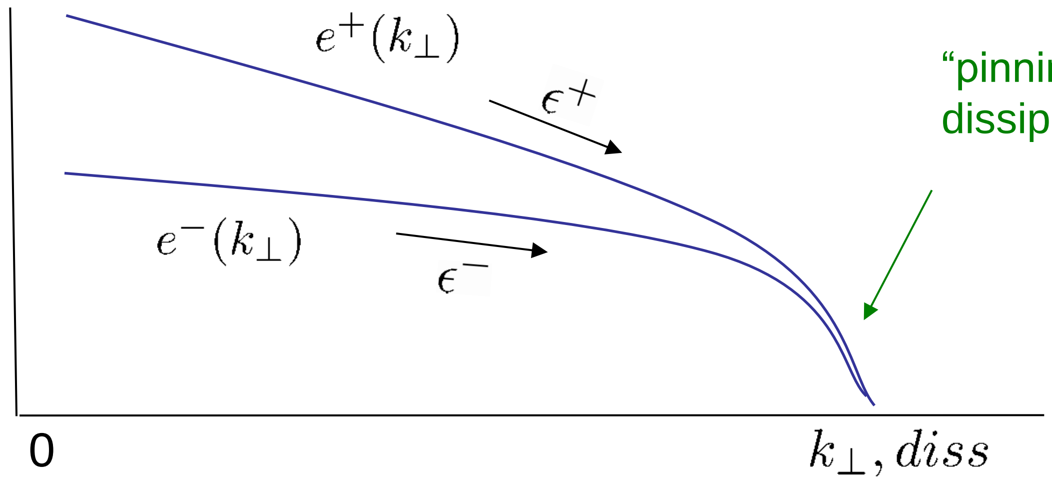
The energy spectra (log-log plot)



"pinning" at the dissipation scale

Imbalanced weak MHD turbulence

(where problems begin)

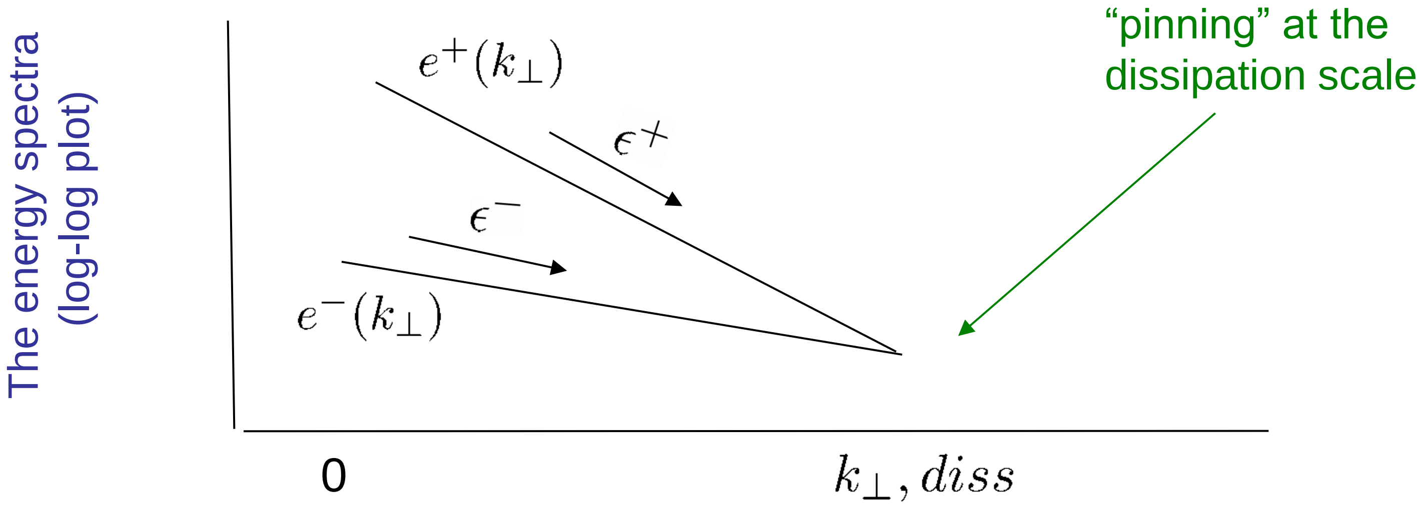
$$e^+(k_z, k_\perp) = g^+(k_z) k_\perp^{-3-\alpha}$$
$$e^-(k_z, k_\perp) = g^-(k_z) k_\perp^{-3+\alpha}$$

$-1 < \alpha < 1$

$$\alpha = f(\epsilon^+/\epsilon^-)$$

The spectra are "pinned" at the dissipation scale.

- If the ratio of the energy fluxes is specified, then the slopes are specified, but the amplitudes depend on the dissipation scale, or on the Re number.



Imbalanced weak MHD turbulence

(where problems begin)

$$e^+(k_z, k_\perp) = g^+(k_z) k_\perp^{-3-\alpha}$$
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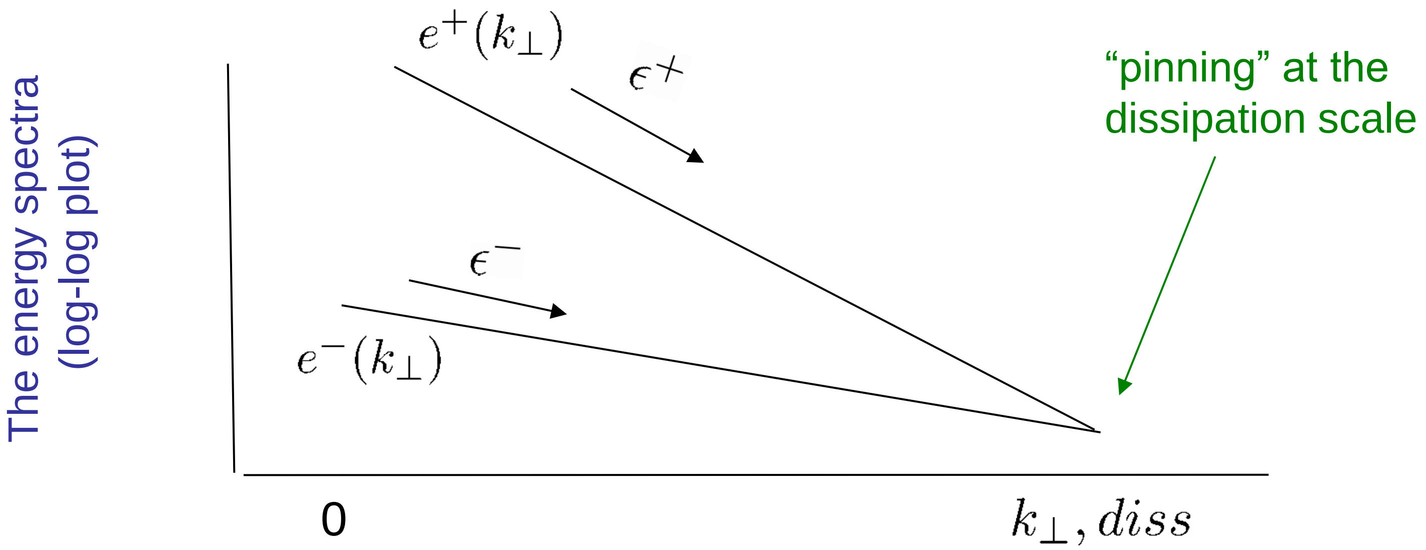
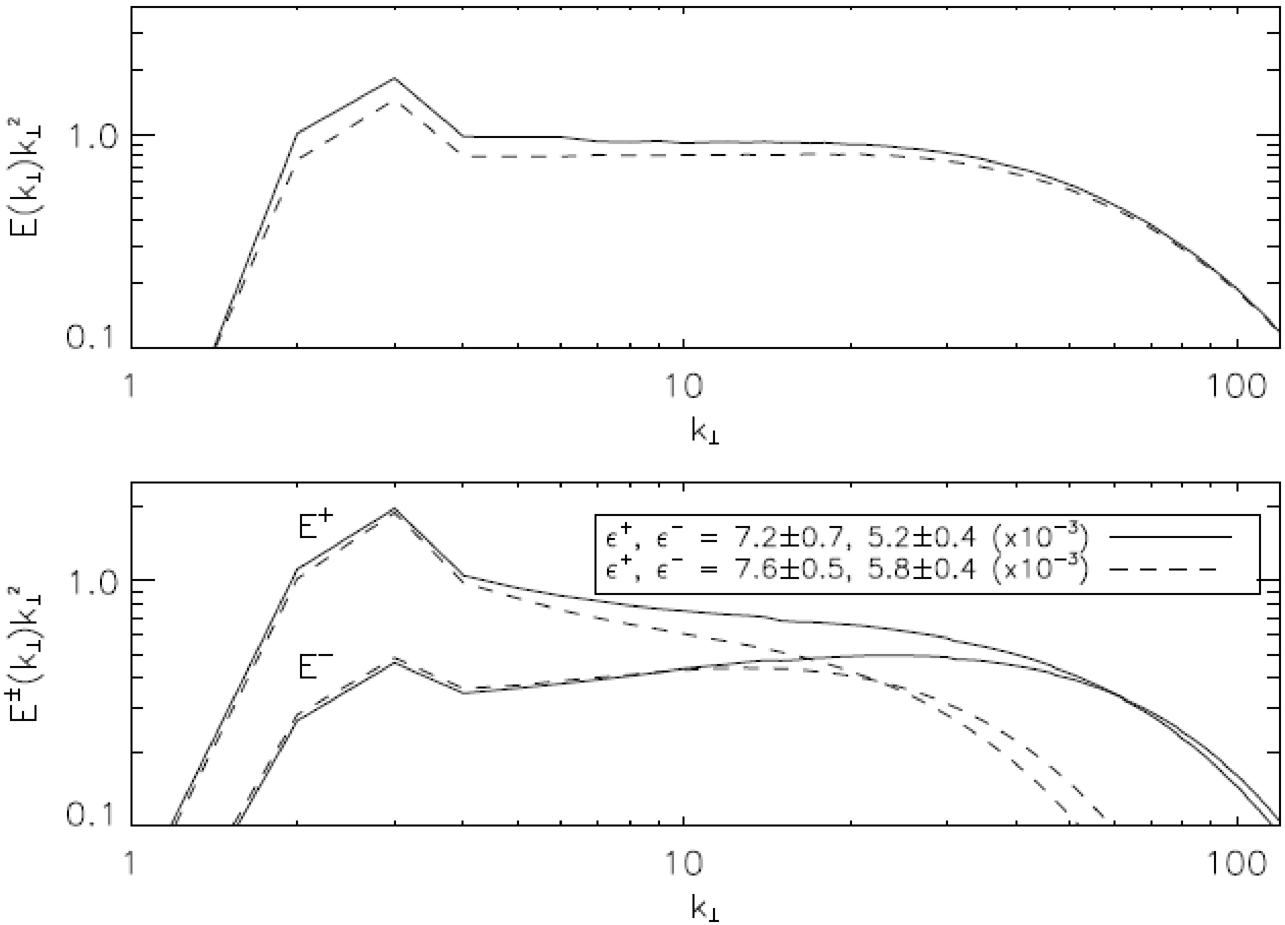


Imbalanced weak MHD turbulence: Numerical results



Balanced

Imbalanced

[SB & Perez (2009)]

Residual energy in weak MHD turbulence

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$ √

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$ √

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = q^r(k_{||}, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$ $\neq 0$

since the waves
are not independent!

$$\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle$$

What is the equation for the residual energy?

SB & Perez PRL 2009

Residual energy in weak MHD turbulence

- Waves are almost independent –
  one would not expect any residual energy!
- Analytically tractable:

$$\partial_t q^r = 2ik_{\|} v_A q^r - \gamma_k q^r +$$
$$+\int R_{k,pq}\{e^+(\mathbf{q})\,[e^-(\mathbf{p})-e^-(\mathbf{k})]+e^-(\mathbf{q})\,[e^+(\mathbf{p})-e^+(\mathbf{k})]\}\delta(q_{\|})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3p\,d^3q$$

where: $R_{k,pq} = (\pi v_A/2)(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2(\mathbf{k}_\perp \cdot \mathbf{p}_\perp)(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)/(k_\perp^2 p_\perp^2 q_\perp^2)$

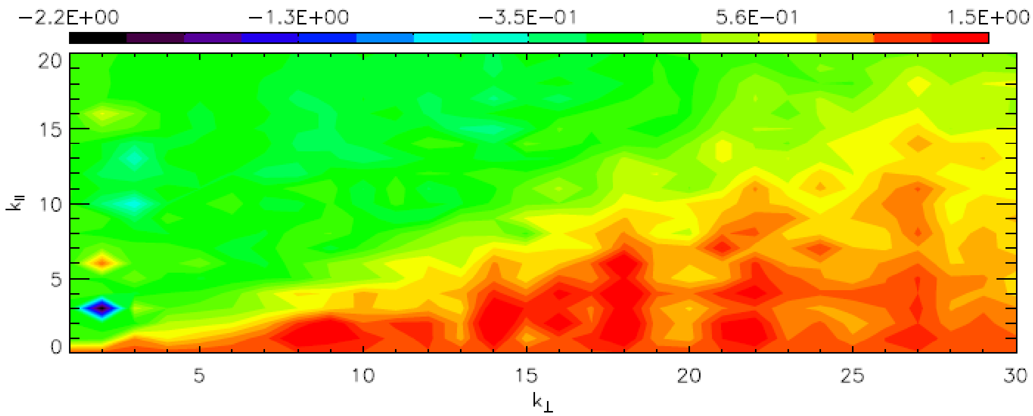
Conclusions:

- *Residual energy is always generated by interacting waves!*
- $\int \ldots < 0$, *so the residual energy is negative:*
  magnetic energy dominates!

Y. Wang, S. B. & J. C. Perez (2011)
S.B, J. C. Perez & V. Zhdankin (2011)

Residual energy in weak MHD turbulence

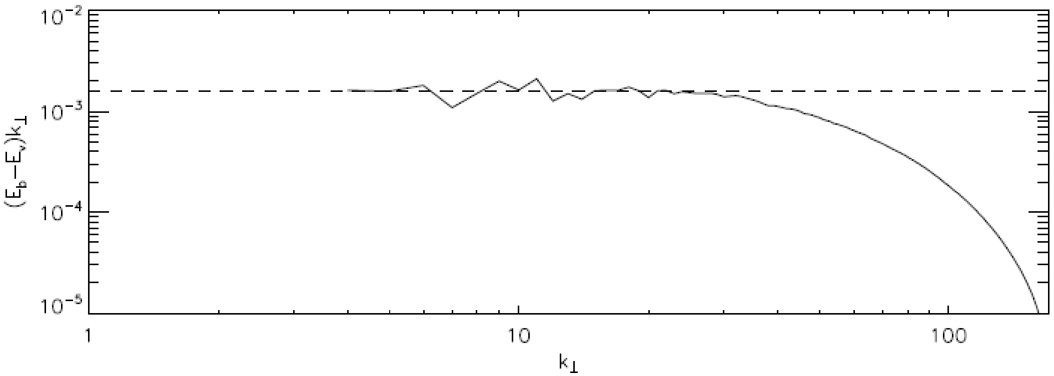
$$e^r(k) = Re\langle z^+(k) \cdot z^-(k)\rangle \propto -\epsilon^2 k_\perp^{-2} \Delta(k_{||})$$



$\Delta(k_{||})$ is concentrated at

$$k_{||} < C\epsilon^2 k_\perp$$

"condensate"



$$E^R(k_\perp) = \int e^r(k_{||}, k_\perp)\, dk_{||} \propto k_\perp^{-1}$$

spectrum of condensate

Y. Wang, S. B. & J. C. Perez (2011)
S.B, J. C. Perez & V. Zhdankin (2011)

Residual energy in weak MHD turbulence

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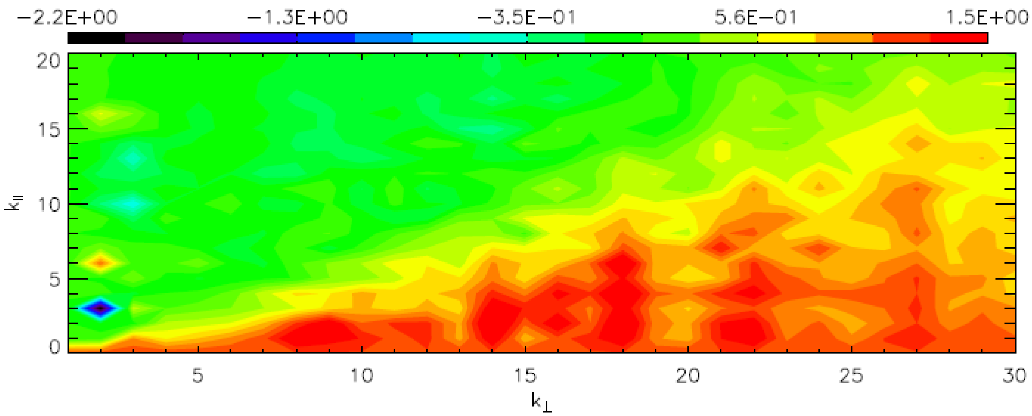

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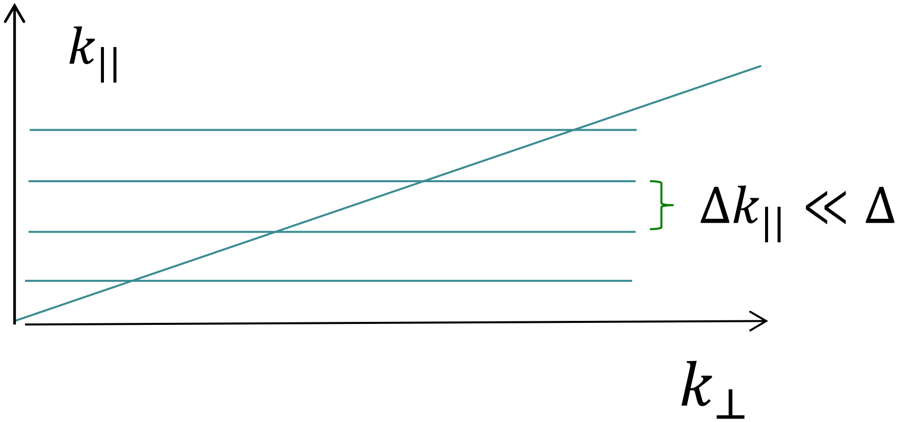
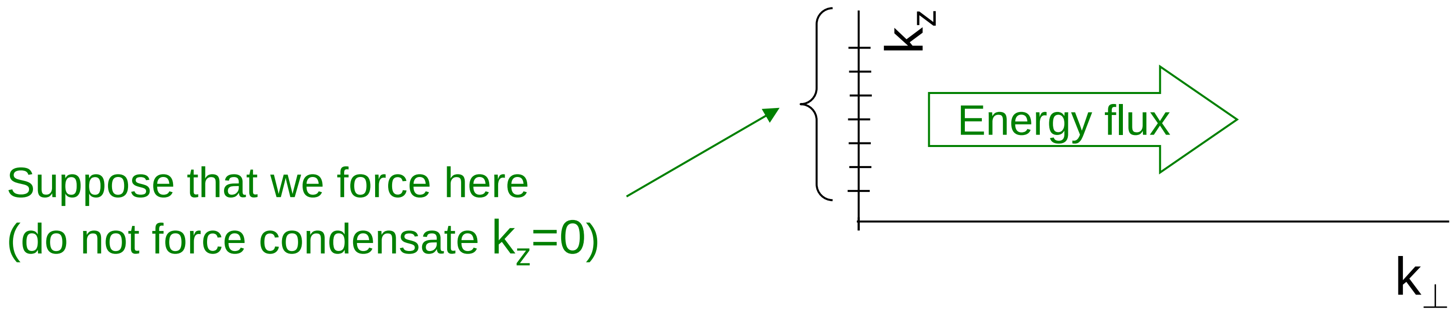
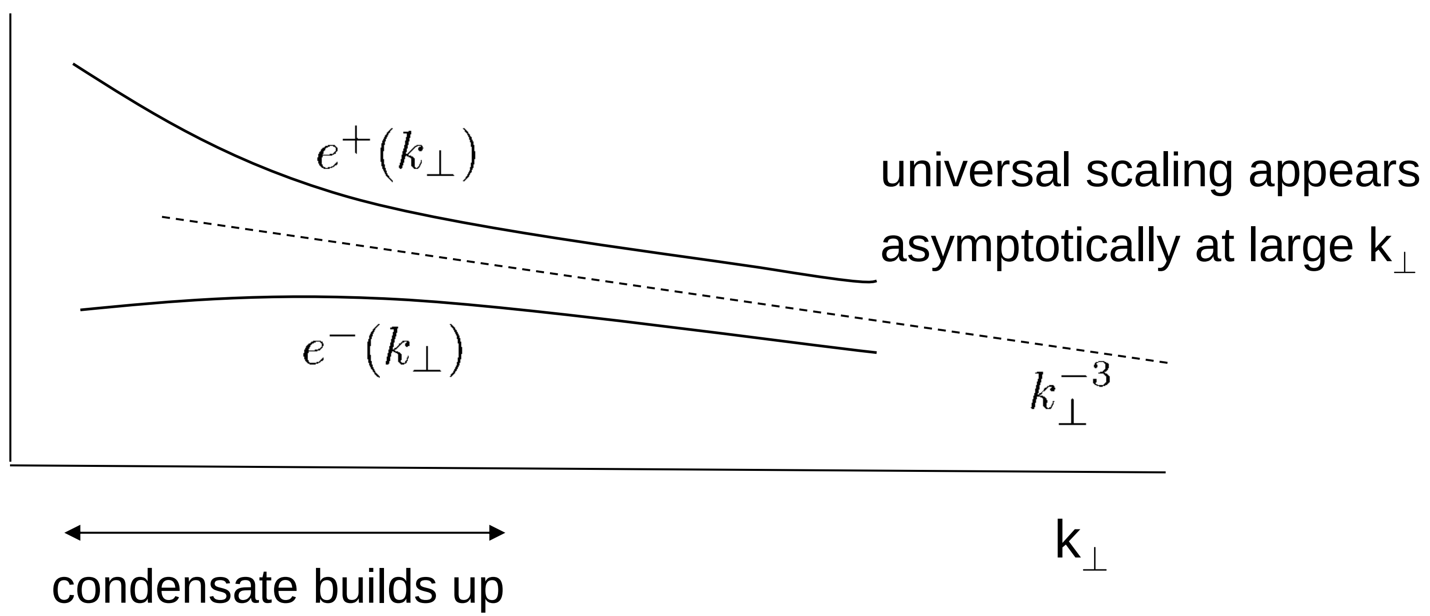


Structure in k-space
must be resolved

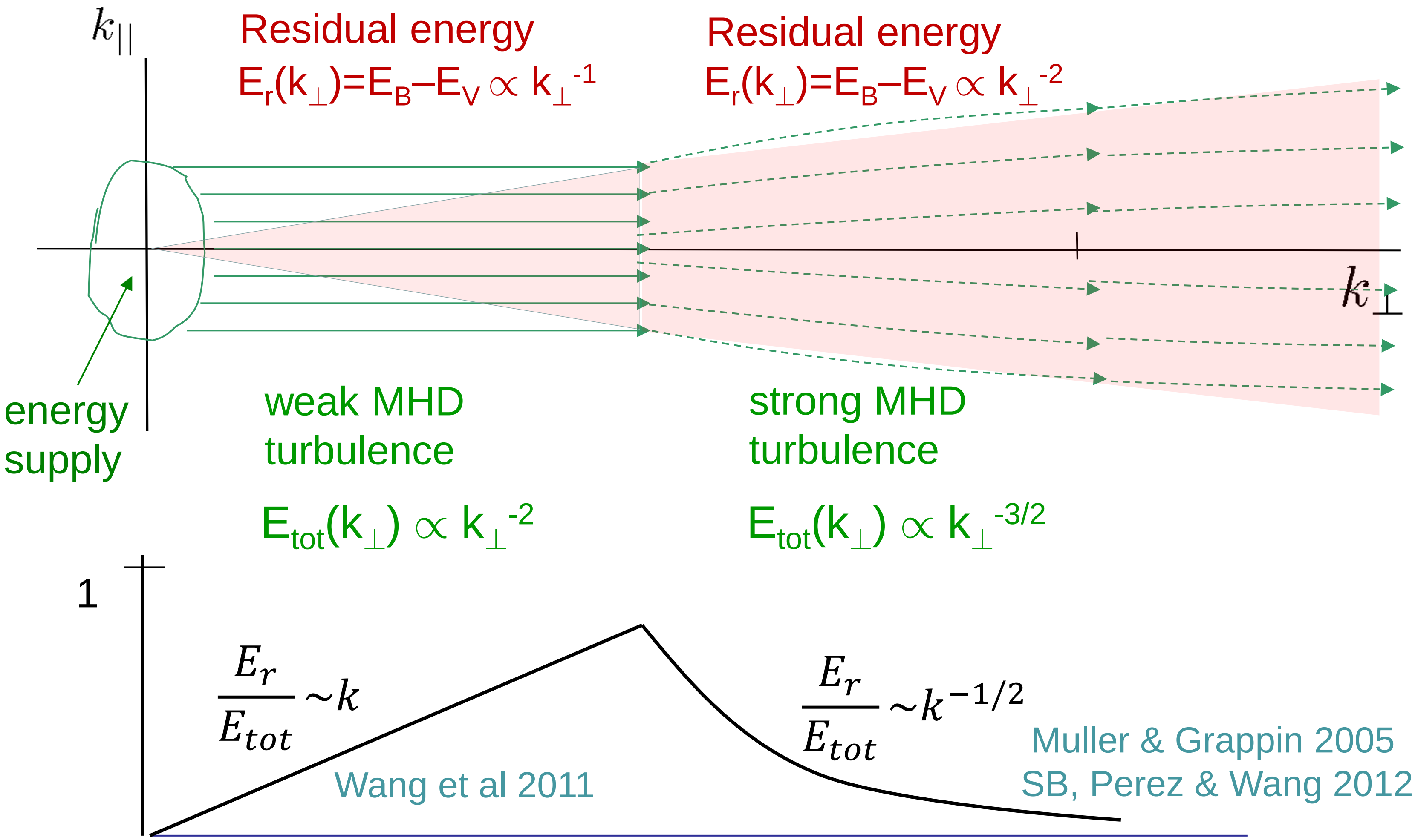
Role of condensate in imbalanced turbulence





$$E^{\pm}(k_\perp) \propto k_\perp^{-2}$$

Universal picture of MHD turbulence



Summary

- Weakly interacting Alfven waves spontaneously generates a condensate of energy (real and residual) at small $k_{||}$.

- The condensate – a large-scale slowly evolving structure -- is a strongly nonlinear object. Waves are scattered by this structure. The wave spectrum is

$$E^{\pm}(k_{\perp}) \propto k_{\perp}^{-2}$$

- Especially important for imbalanced turbulence.