Session 2: **Anisotropy** and Kinetic Effects
(Tues, May 21, 9-10:30 am and 11am-12:30 pm)

Anisotropy in ideal MHD turbulence due to rotation and/or a mean magnetic field

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Overview

- MHD Turbulence is inherently anisotropic
- MHD and HD turbulence fundamentally differ
- Real MHD turbulence ~ inertial & small scales
- Ideal MHD turbulence provides some guidance

$\rightarrow$ Limit of hyper-dissipation as $n \rightarrow \infty$

$\rightarrow$ Quasi-equilibrium theory for large-scales

$\rightarrow$ Anisotropy through broken symmetry

$\rightarrow$ $\mathbf{\Omega}_o$ or $\mathbf{B}_o$ give further structure

Basic Equations

$$\boxed{\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial t} &= \nabla \times [\mathbf{u} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega}_\mathrm{o}) + \mathbf{j} \times (\mathbf{b} + \mathbf{B}_\mathrm{o})] + \nu \nabla^2 \boldsymbol{\omega}, \\ \frac{\partial \mathbf{b}}{\partial t} &= \nabla \times [\mathbf{u} \times (\mathbf{b} + \mathbf{B}_\mathrm{o})] + \eta \nabla^2 \mathbf{b}. \end{aligned}}$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad \mathbf{j} = \nabla \times \mathbf{b}.$$

$$\boldsymbol{\Omega}_\mathrm{o} = \Omega_\mathrm{o} \hat{\mathbf{z}} \text{ and } \mathbf{B}_\mathrm{o} = B_\mathrm{o} \hat{\mathbf{x}}$$

Five Cases of Ideal MHD Turbulence ($\nu = \eta = 0$)

Case	Mean field	Rotation	Invariants
I	$\mathbf{B}_o = \mathbf{0}$	$\boldsymbol{\Omega}_o = \mathbf{0}$	$E,\ H_C,\ H_M$
II	$\mathbf{B}_o \neq \mathbf{0}$	$\boldsymbol{\Omega}_o = \mathbf{0}$	$E,\ H_C$
III	$\mathbf{B}_o = \mathbf{0}$	$\boldsymbol{\Omega}_o \neq \mathbf{0}$	$E,\ H_M$
IV	$\mathbf{B}_o \neq \mathbf{0}$	$\boldsymbol{\Omega}_o = \sigma \mathbf{B}_o$	$E,\ H_P$
V	$\mathbf{B}_o \neq \mathbf{0}$	$\boldsymbol{\Omega}_o \times \mathbf{B}_o \neq \mathbf{0}$	E

$H_P = H_C - \sigma H_M$, where σ is defined by the relation $\boldsymbol{\Omega}_o = \sigma \mathbf{B}_o$.

Fourier Transform

Fourier transformation from **x**-space to **k**-space and back.

$$\begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \mathbf{b}(\mathbf{x},t) \end{bmatrix} = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \begin{bmatrix} \tilde{\mathbf{u}}(\mathbf{k},t) \\ \tilde{\mathbf{b}}(\mathbf{k},t) \end{bmatrix} e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\begin{bmatrix} \tilde{\mathbf{u}}(\mathbf{k},t) \\ \tilde{\mathbf{b}}(\mathbf{k},t) \end{bmatrix} = \frac{1}{N^{3/2}} \sum_{\mathbf{x}} \begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \mathbf{b}(\mathbf{x},t) \end{bmatrix} e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

$$\begin{aligned} \tilde{\mathbf{u}}(\mathbf{k}) &= \tilde{u}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{u}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}), \\ \tilde{\mathbf{b}}(\mathbf{k}) &= \tilde{b}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{b}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}). \end{aligned}$$

Statistical analysis done on Fourier modes in **k**-space.

Numerical solution by Fourier spectral transform method.

Statistical Mechanics of Ideal MHD Turbulence

(Frisch, et al., *JFM*, 1975)

$$D = \frac{1}{Z}\exp(-\alpha\widehat{E} - \beta\widehat{H}_C - \gamma\widehat{H}_M)$$

$$Z = \int_\Gamma \exp(-\alpha\widehat{E} - \beta\widehat{H}_C - \gamma\widehat{H}_M)d\Gamma$$

Quadratic invariants

$$\widehat{E} = \frac{1}{N^3}\sum_{\mathbf{k}'}\sum_{n=1}^{2}\left[|\tilde{u}_n(\mathbf{k})|^2 + |\tilde{b}_n(\mathbf{k})|^2\right],$$

$$\widehat{H}_C = \frac{1}{N^3}\sum_{\mathbf{k}'}\sum_{n=1}^{2}\left[\tilde{u}_n^R(\mathbf{k})\tilde{b}_n^R(\mathbf{k}) + \tilde{u}_n^I(\mathbf{k})\tilde{b}_n^I(\mathbf{k})\right],$$

$$\widehat{H}_M = \frac{2}{N^3}\sum_{\mathbf{k}'}\frac{1}{k}\left[\tilde{b}_1^R(\mathbf{k})\tilde{b}_2^I(\mathbf{k}) - \tilde{b}_2^R(\mathbf{k})\tilde{b}_1^I(\mathbf{k})\right].$$

Ideal MHD Modal PDF

$$D = \prod_{\mathbf{k}'} D(\mathbf{k}), \qquad D(\mathbf{k}) = Z^{-1}(\mathbf{k}) \exp[-\tilde{Y}^{\dagger}(\mathbf{k}) M_k \tilde{Y}(\mathbf{k})].$$

$$\tilde{Y}(\mathbf{k}) = \begin{bmatrix} \tilde{u}_1(\mathbf{k}) \\ \tilde{u}_2(\mathbf{k}) \\ \tilde{b}_1(\mathbf{k}) \\ \tilde{b}_2(\mathbf{k}) \end{bmatrix}, \quad M_k = \begin{bmatrix} \widehat{\alpha} & 0 & \widehat{\beta}/2 & 0 \\ 0 & \widehat{\alpha} & 0 & \widehat{\beta}/2 \\ \widehat{\beta}/2 & 0 & \widehat{\alpha} & -\mathrm{i}\widehat{\gamma}/k \\ 0 & \widehat{\beta}/2 & \mathrm{i}\widehat{\gamma}/k & \widehat{\alpha} \end{bmatrix}.$$

Ideal MHD Eigenanalysis

$$\begin{aligned}
\tilde{v}_1(\mathbf{k}) &= \widehat{\beta}_s \zeta_k^- \tilde{u}_+(\mathbf{k}) - \zeta_k^+ \tilde{b}_+(\mathbf{k}), \\
\tilde{v}_2(\mathbf{k}) &= \widehat{\beta}_s \zeta_k^- \tilde{u}_-(\mathbf{k}) + \zeta_k^+ \tilde{b}_-(\mathbf{k}), \\
\tilde{v}_3(\mathbf{k}) &= \widehat{\beta}_s \zeta_k^+ \tilde{u}_+(\mathbf{k}) + \zeta_k^- \tilde{b}_+(\mathbf{k}), \\
\tilde{v}_4(\mathbf{k}) &= -\widehat{\beta}_s \zeta_k^+ \tilde{u}_-(\mathbf{k}) + \zeta_k^- \tilde{b}_-(\mathbf{k}).
\end{aligned}$$

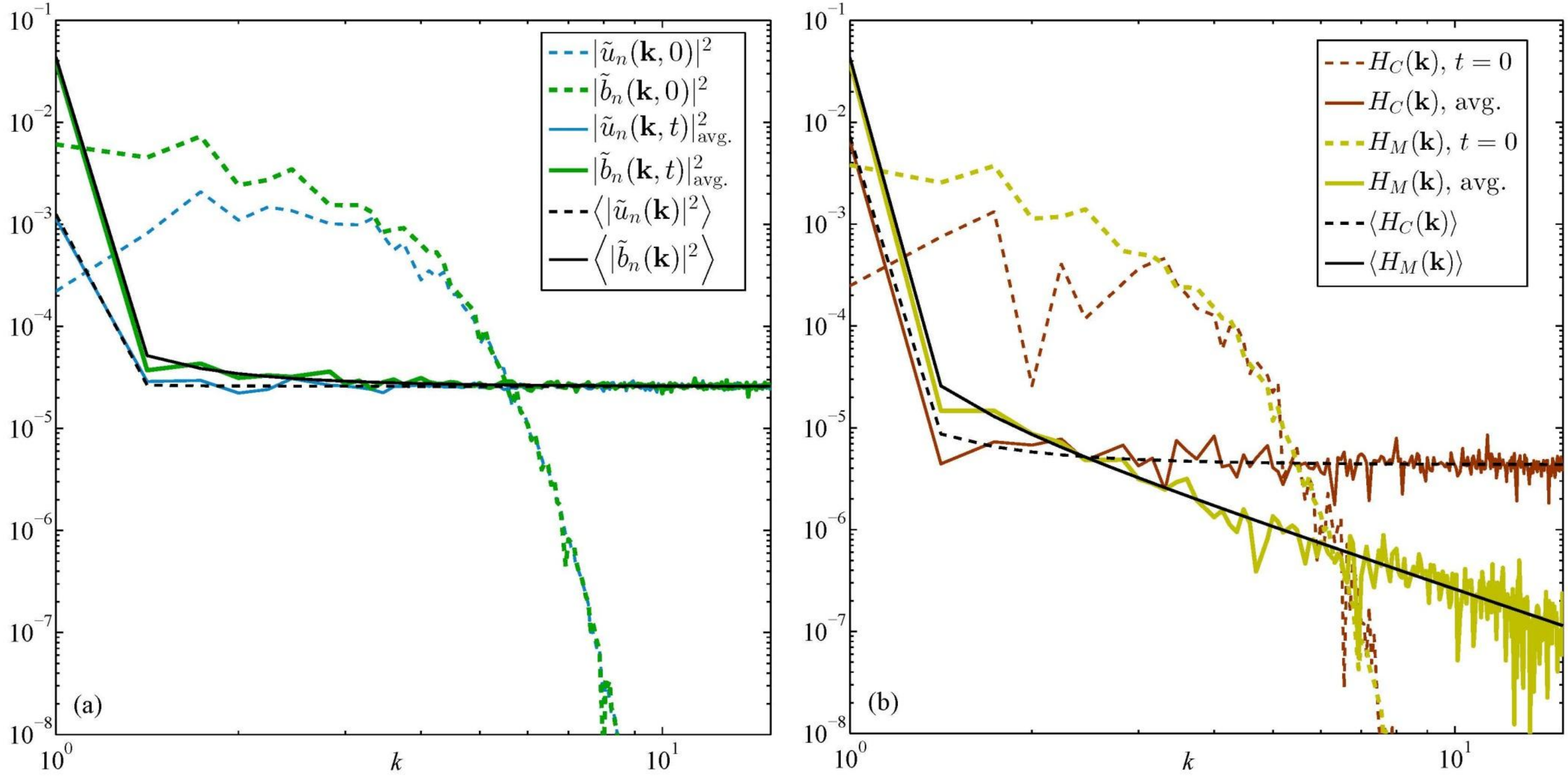
$$\tilde{u}_\pm(\mathbf{k}) = \tilde{u}_1(\mathbf{k}) \pm \mathrm{i}\tilde{u}_2(\mathbf{k}), \qquad \tilde{b}_\pm(\mathbf{k}) = \tilde{b}_1(\mathbf{k}) \pm \mathrm{i}\tilde{b}_2(\mathbf{k}).$$

$$\zeta_k^\pm = \frac{1}{2}\sqrt{1 \pm \frac{\widehat{\gamma}}{k\widehat{\eta}_k}}; \qquad \widehat{\eta}_k = \sqrt{\widehat{\beta}^2 + \frac{\widehat{\gamma}^2}{k^2}}.$$

$$\widehat{\lambda}_k^{(1)} = \widehat{\alpha} - \frac{1}{2}(\widehat{\eta}_k + \widehat{\gamma}/k), \qquad \widehat{\lambda}_k^{(2)} = \widehat{\alpha} + \frac{1}{2}(\widehat{\eta}_k + \widehat{\gamma}/k),$$

$$\widehat{\lambda}_k^{(3)} = \widehat{\alpha} + \frac{1}{2}(\widehat{\eta}_k - \widehat{\gamma}/k), \qquad \widehat{\lambda}_k^{(4)} = \widehat{\alpha} - \frac{1}{2}(\widehat{\eta}_k - \widehat{\gamma}/k).$$

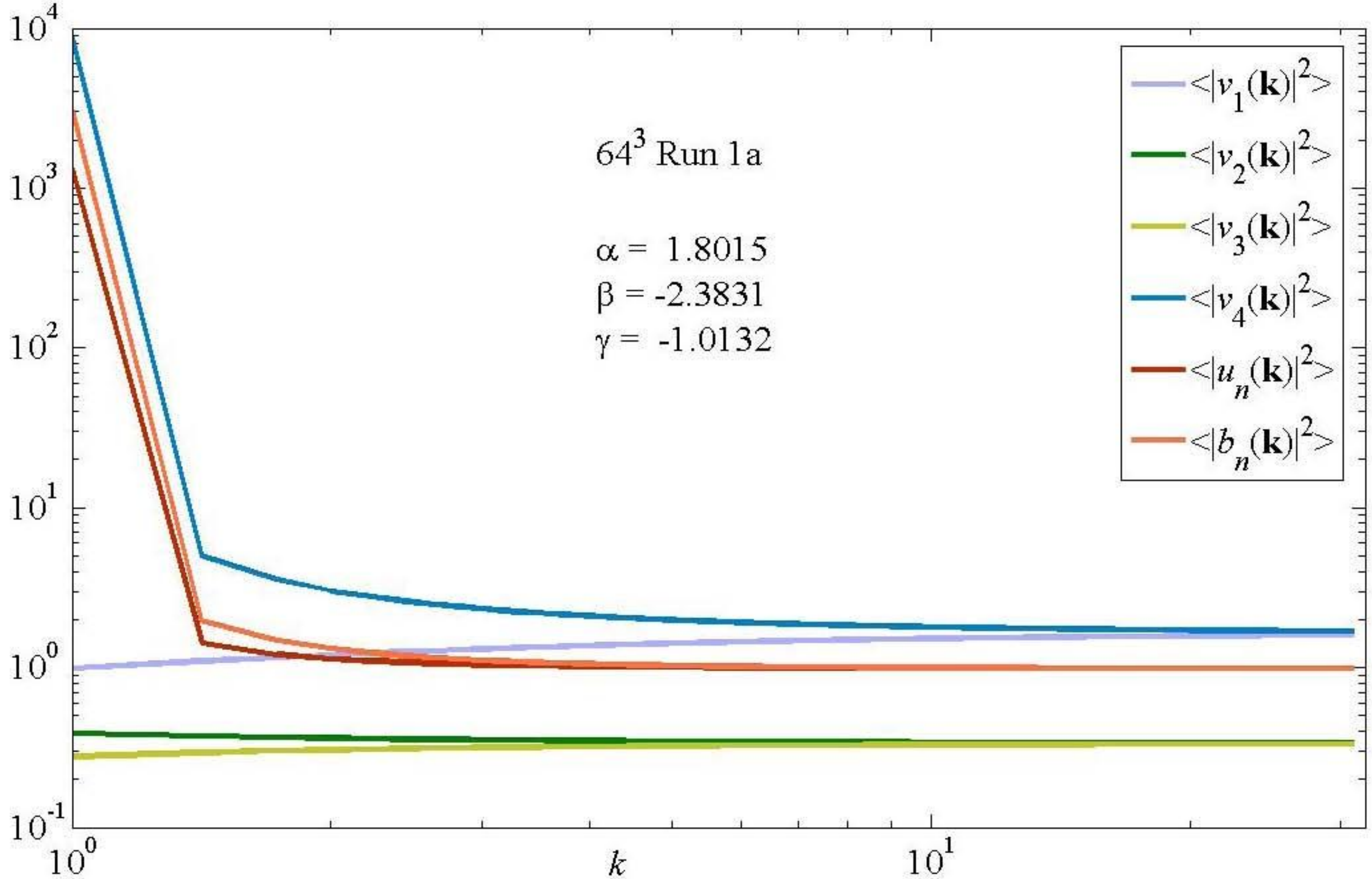
Ideal MHD Spectra



64^3; E = 1.000, H_C = 0.3480, H_M = 0.09197; $\Delta t = 10^{-3}$; t = 0 to 200.

[$u_n(\mathbf{k})$ and $b_n(\mathbf{k})$ come from a Craya decomposition of $\mathbf{u}(\mathbf{k})$ and $\mathbf{b}(\mathbf{k})$.]

Eigenvariable Spectra: $|v_n(\mathbf{k})|^2 = 1/\lambda_k^{(n)}$

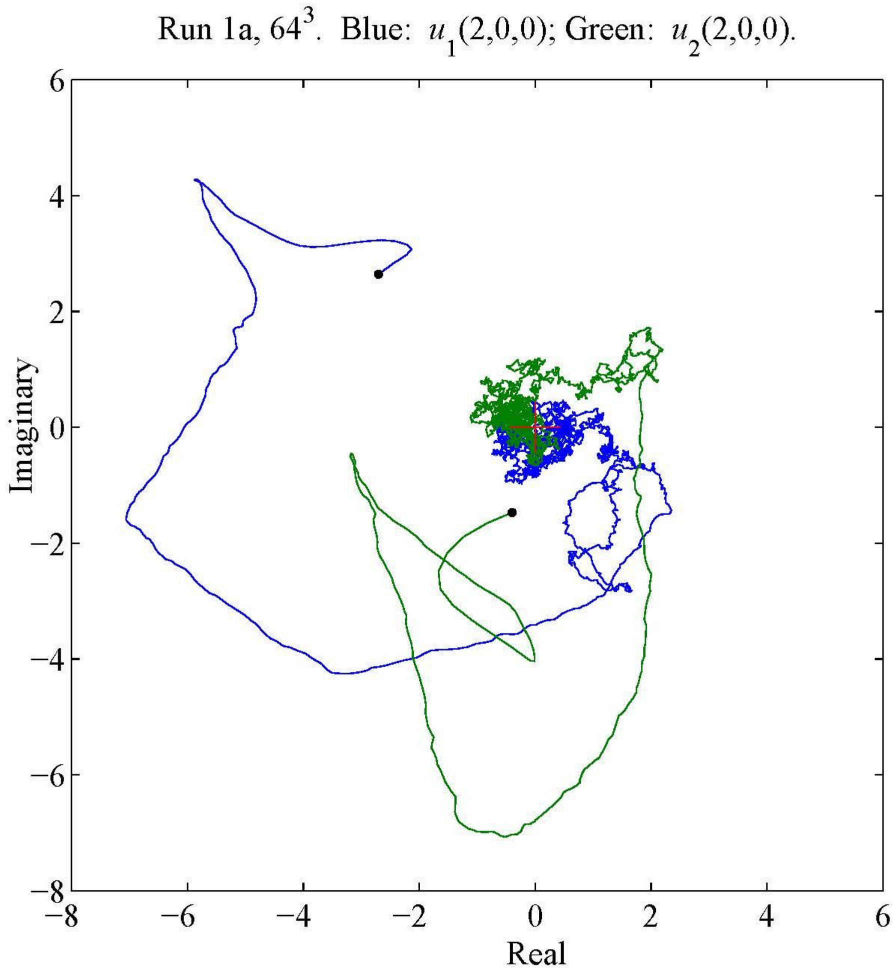


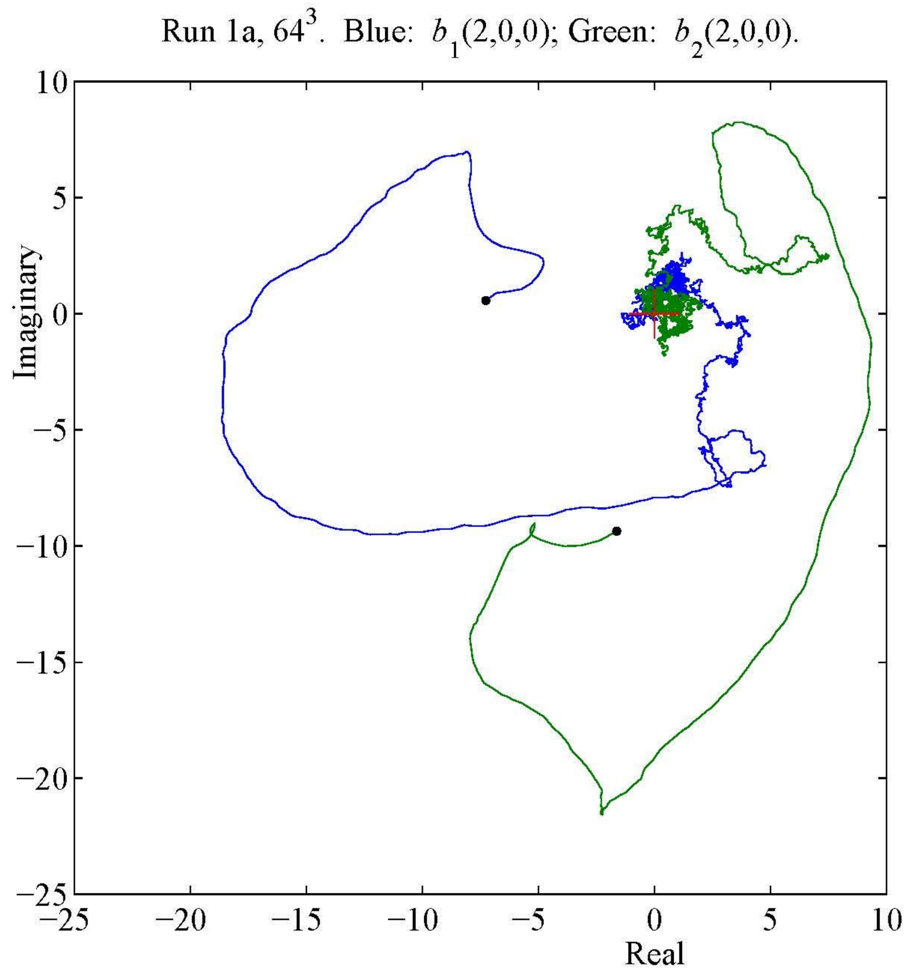
$|v_4(\mathbf{k})|^2 \sim O(E)$ for $k^2 = 1$, but other $|v_n(\mathbf{k})|^2 \sim O(E/N^3)$.

Expectation: Zero-mean Random Variables

$$\tilde{\mathbf{u}}(\mathbf{k}) \;=\; \tilde{u}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{u}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}),$$

$$\tilde{\mathbf{b}}(\mathbf{k}) \;=\; \tilde{b}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{b}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}).$$

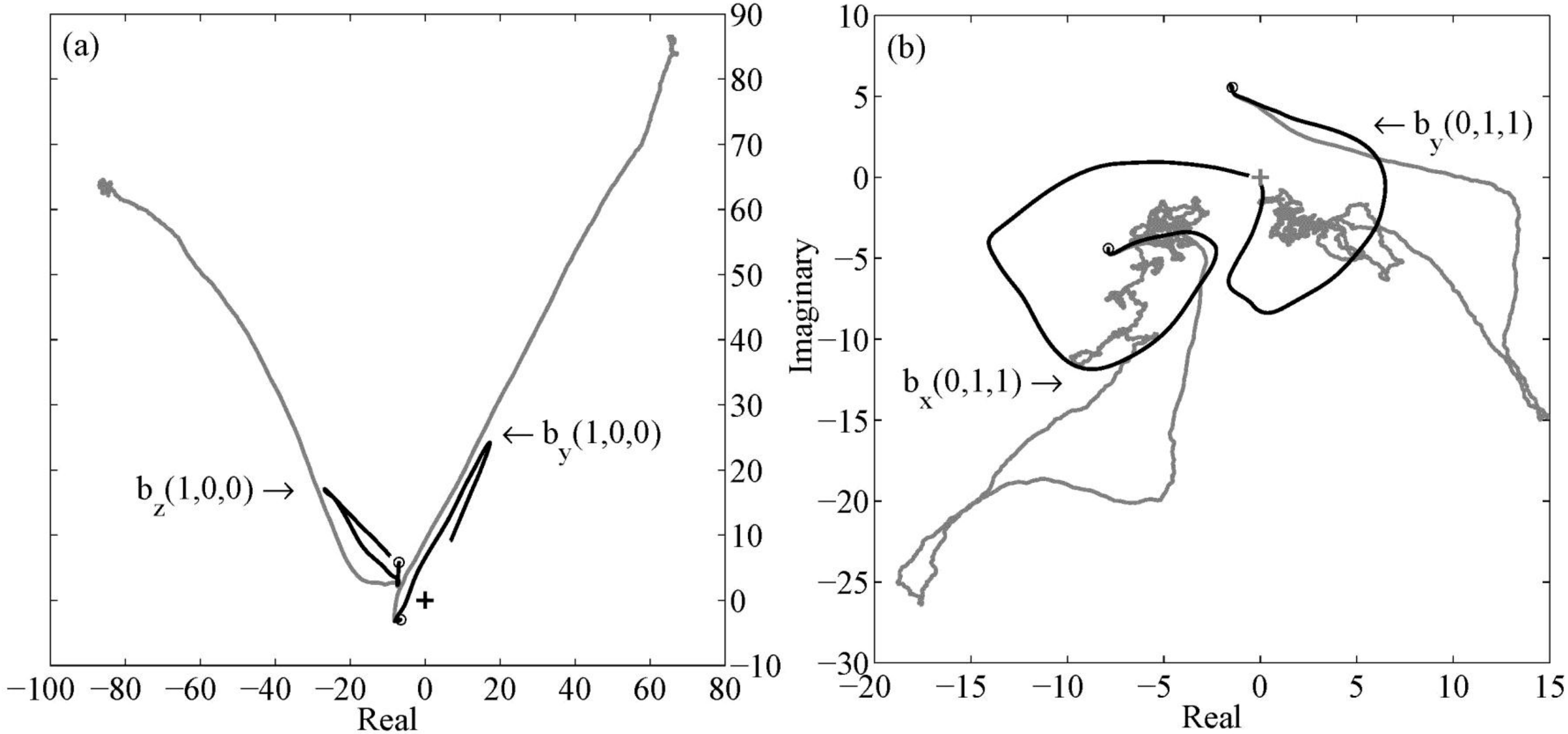




PDF is product of modal PDFs: $D = \prod_{\mathbf{k}'} D(\mathbf{k})$

Reality: Broken Ergodicity & Symmetry

64^3 runs 1a (gray: $\nu = \eta = 0$) and 1 (black: $\nu = \eta = 0.004$)



Low-*k* modes don't behave as expected.

This gives large-scale anisotropy in MHD turbulence .

Broken Ergodicity

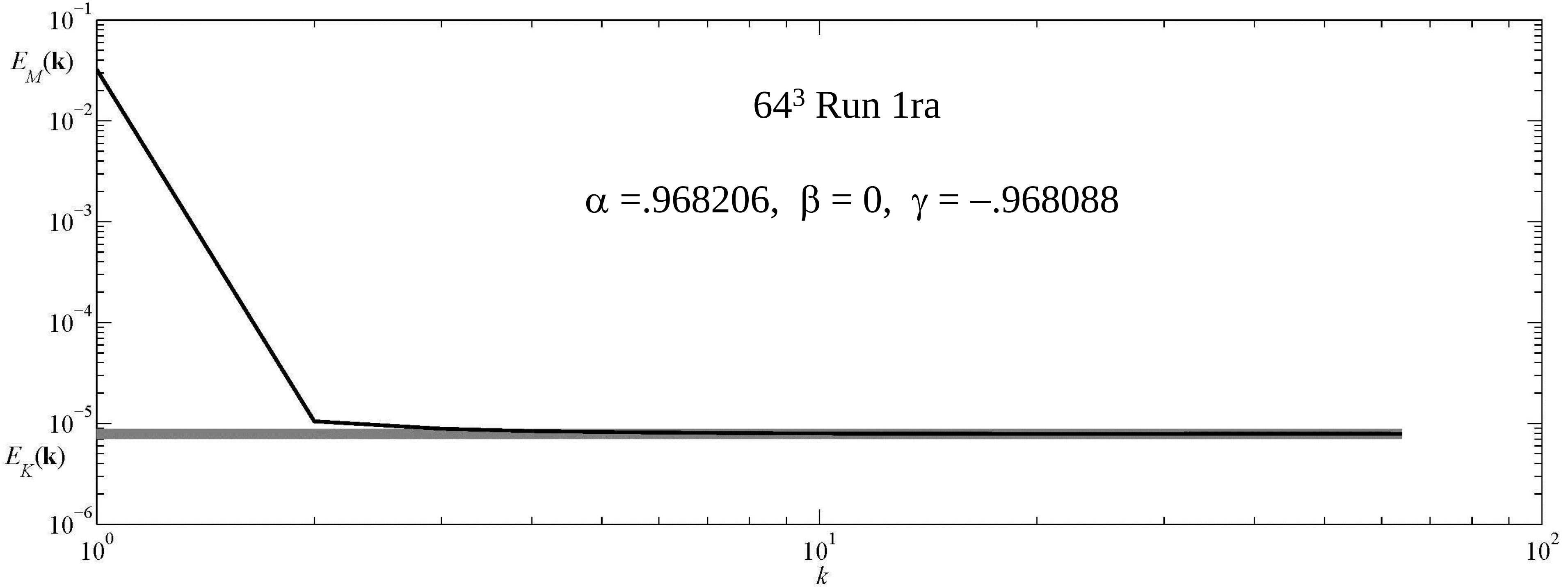
Statistical theory tells us that $\tilde{v}_4(\hat{\mathbf{k}}),\ |\hat{\mathbf{k}}|=1,$ is very large, so

$$
\begin{aligned}
0 &\approx -\zeta_1^- \tilde{u}_+(\hat{\mathbf{k}}) - \zeta_1^+ \tilde{b}_+(\hat{\mathbf{k}}), \\
0 &\approx -\zeta_1^- \tilde{u}_-(\hat{\mathbf{k}}) + \zeta_1^+ \tilde{b}_-(\hat{\mathbf{k}}), \\
0 &\approx -\zeta_1^+ \tilde{u}_+(\hat{\mathbf{k}}) + \zeta_1^- \tilde{b}_+(\hat{\mathbf{k}}), \\
\tilde{v}_4(\hat{\mathbf{k}}) &= +\zeta_1^+ \tilde{u}_-(\hat{\mathbf{k}}) + \zeta_1^- \tilde{b}_-(\hat{\mathbf{k}}).
\end{aligned}
$$

$$
\tilde{u}_2(\hat{\mathbf{k}}) \approx i\tilde{u}_1(\hat{\mathbf{k}}), \quad \tilde{b}_2(\hat{\mathbf{k}}) \approx i\tilde{b}_1(\hat{\mathbf{k}}), \quad \tilde{v}_4(\hat{\mathbf{k}}) \approx -\tilde{b}_1(\hat{\mathbf{k}})/\zeta_1^- \approx -\tilde{u}_1(\hat{\mathbf{k}})/\zeta_1^+.
$$

$$
\frac{d\tilde{v}_4(\hat{\mathbf{k}})}{dt} \cong 0 \quad \Rightarrow \quad \tilde{v}_4(\hat{\mathbf{k}}) \text{ become quasistationary.}
$$

Rotation: $\mathbf{\Omega}_o \neq \mathbf{0} \rightarrow \beta = 0.$



Linear eigenmodes ~ $\exp(\pm i2\mathbf{k}\cdot\mathbf{\Omega}_o t)$

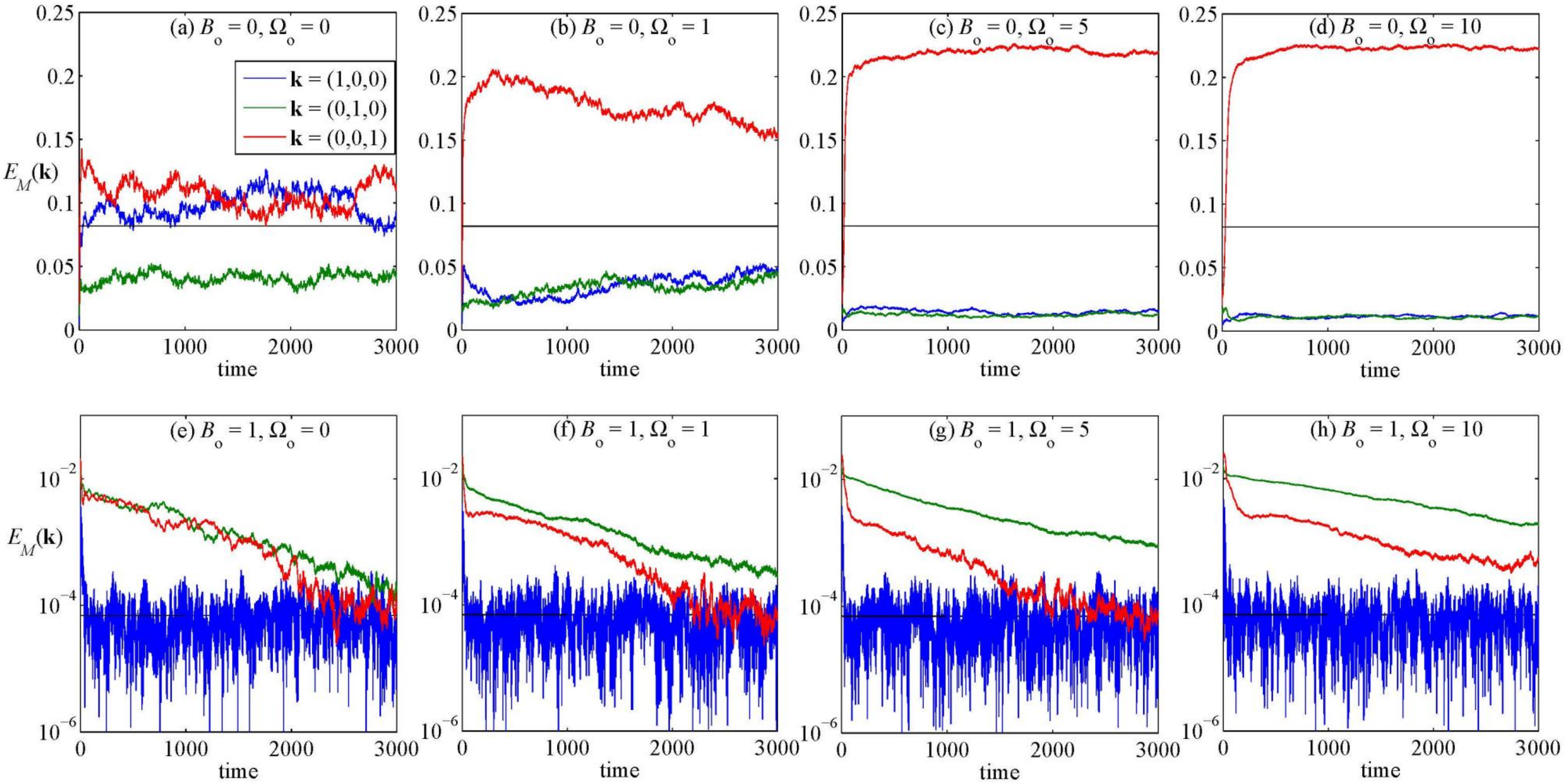
$$\tilde{v}_1(\mathbf{k},t) = \frac{1}{\sqrt{2}}[\tilde{u}_1(\mathbf{k},t) + i\tilde{u}_2(\mathbf{k},t)], \quad \tilde{v}_2(\mathbf{k},t) = \frac{i}{\sqrt{2}}[\tilde{u}_1(\mathbf{k},t) - i\tilde{u}_2(\mathbf{k},t)],$$

Nonlinear eigenmodes

$$\tilde{v}_3(\mathbf{k},t) = \frac{1}{\sqrt{2}}[\tilde{b}_1(\mathbf{k},t) + i\tilde{b}_2(\mathbf{k},t)], \quad \tilde{v}_4(\mathbf{k},t) = \frac{i}{\sqrt{2}}[\tilde{b}_1(\mathbf{k},t) - i\tilde{b}_2(\mathbf{k},t)].$$

Recent 32^3 Runs: $E_M(\mathbf{k})$ at $k^2 = 1$

$$\boldsymbol{\Omega}_o = \Omega_o \hat{\mathbf{z}} \text{ and } \mathbf{B}_o = B_o \hat{\mathbf{x}}$$



At $t = 0$, $E_K = 0.34000$, $E_M = 0.66000$, $H_C = 0.10459$ and $H_M = 0.25071$.

Effect of Rotation

32^3 Run R5-1, $\boldsymbol{\Omega}_o = \mathbf{0}$;

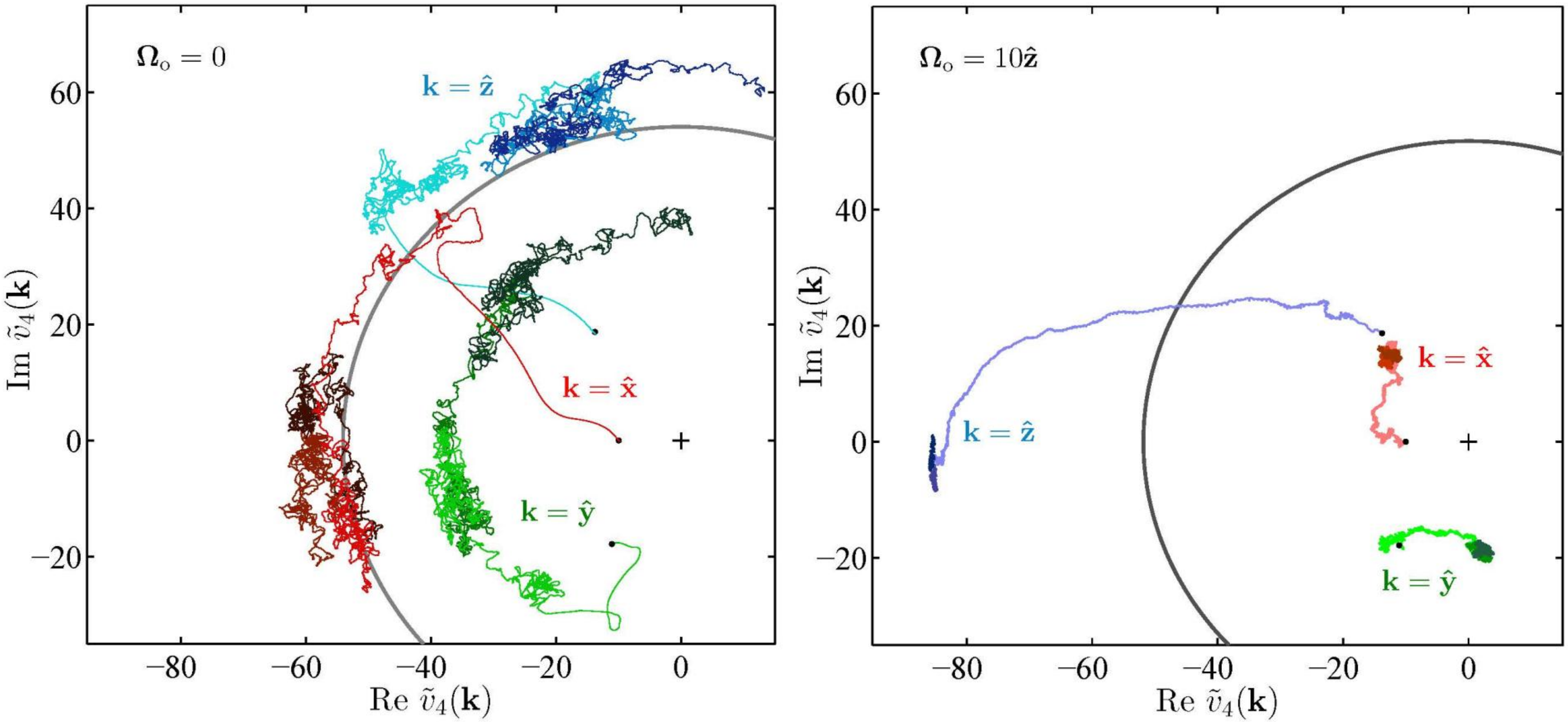
$\alpha = 1.215331$, $\beta = -0.408862$,

$\gamma = -1.180571$

32^3 Run R5-4, $\boldsymbol{\Omega}_o = 10\hat{\mathbf{z}}$;

$\alpha = 1.169767$, $\beta = 0$,

$\gamma = -1.169394$



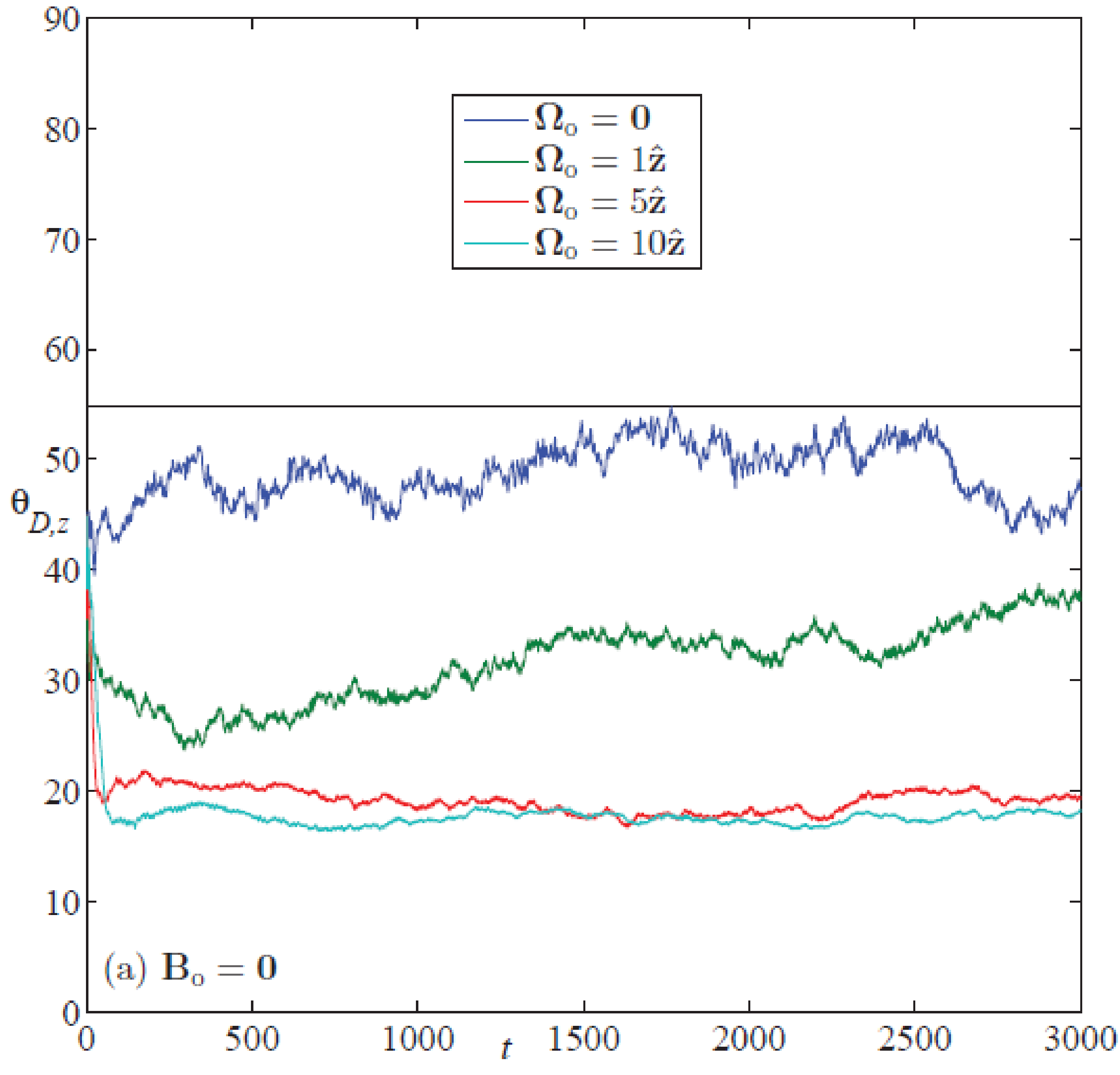
Rotational Anisotropy → Dipole Angle

$$\theta_{D,z} \equiv \tan^{-1}\sqrt{\frac{E_M(\hat{\mathbf{x}}) + E_M(\hat{\mathbf{y}})}{E_M(\hat{\mathbf{z}})}}$$

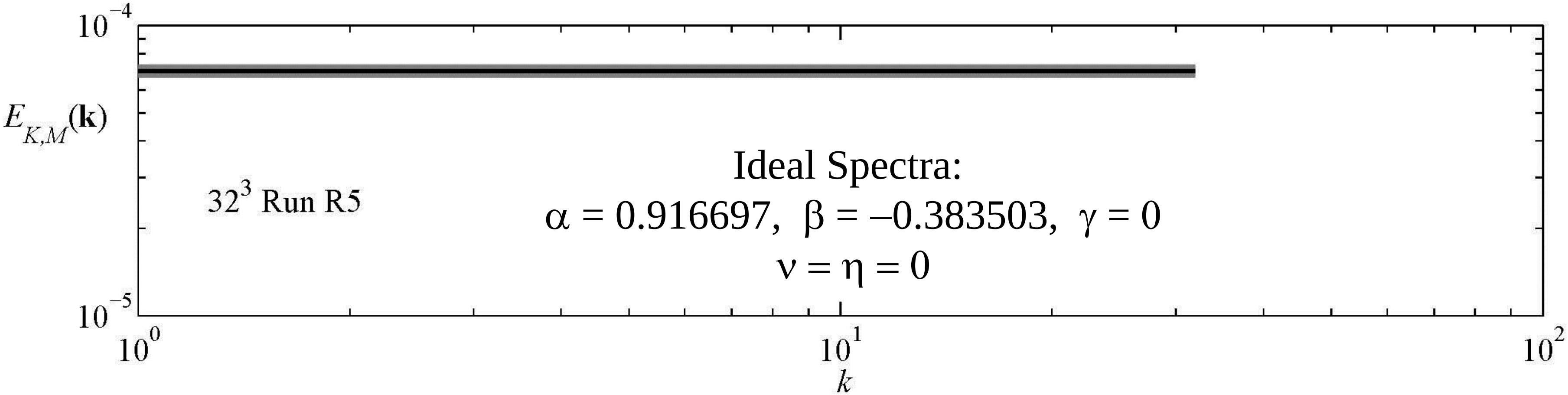
$$\tan^{-1}\sqrt{2} = 54.74 \text{ deg}$$

Dipole angle appears to

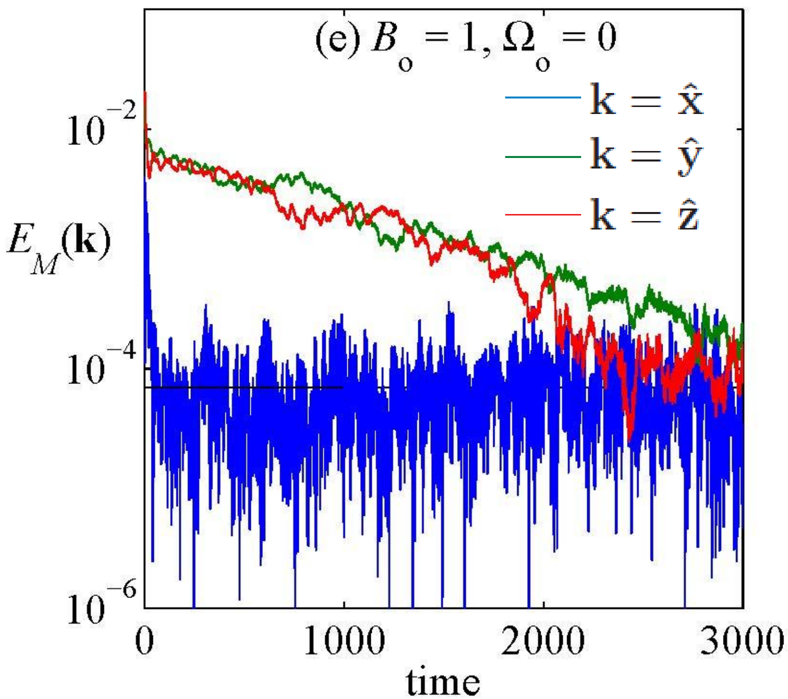
saturate as Ω_o increases.



Mean Magnetic Field: $\mathbf{B}_o \neq \mathbf{0} \rightarrow \gamma = 0$



Ideal MHD



$$\boxed{\mathbf{B}_o = B_o \hat{\mathbf{x}}}$$

Real MHD

Dynamical anisotropy [$E(\mathbf{k}_\perp) > E(\mathbf{k}_\|)$] due to resonant excitation of Alfven waves:

$$\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2,$$

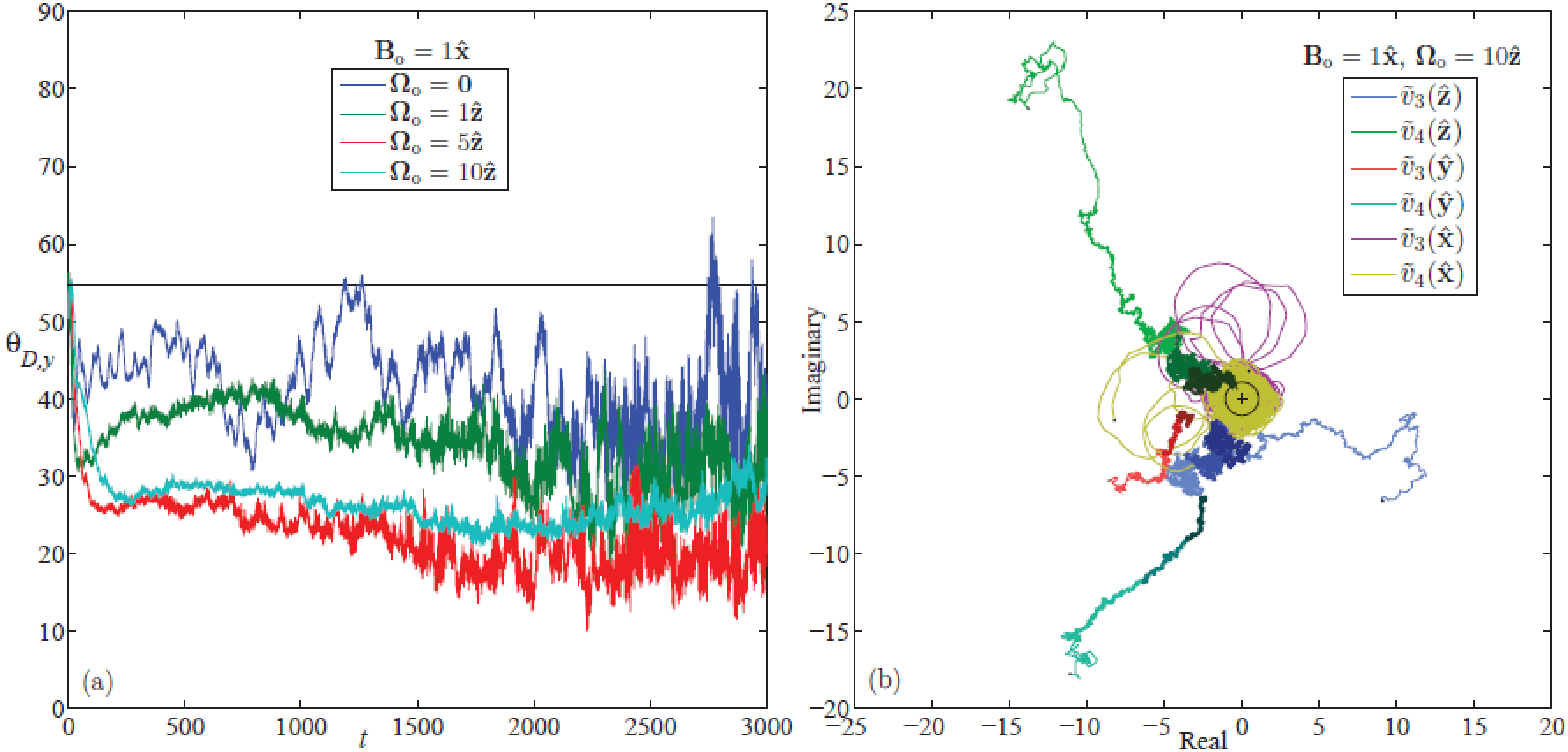
$$\pm\omega(\mathbf{k}_3) = \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2).$$

Above, $\omega(\mathbf{k}) = \mathbf{B}_o \cdot \mathbf{k}$. Thus, either $\omega(\mathbf{k}_1)$ or $\omega(\mathbf{k}_2) = 0$.

Also, anisotropy increases as k increases.

(Shebalin, et al., *JPP*, 1983; Oughton, et al., *JFM*, 1994.)

Rotation and Mean Field Effects



32^3 runs R5 to R8: $t = 0$ to 3000 ($\Delta t = 10^{-3}$); stationarity not yet reached.

Evolution towards equilibrium of $k = 1$ modes $\perp$ to $\mathbf{B}_o$ appears to slow down.

The dipole moment ($k = 1$) appears to line up with $\mathbf{B}_o \times \boldsymbol{\Omega}_o$, in the short-term.

Fluctuations in $\theta_{D,y}$ decrease as $|\boldsymbol{\Omega}_o|$ increases.

Conclusions

- When $B_o = 0$ (*e.g.*, geodynamo), lowest-k modes can have large energies.

  In this case, ideal MHD results appear to carry over to ν, $\eta > 0$.

  Energetic, coherent structures form at the largest scales.

  These are **anisotropic** through broken ergodicity & symmetry.

  When $\Omega_o \neq 0$, dipole alignment occurs: further **anisotropy**.
- When $B_o \neq 0$ (*e.g.*, SW), all modes are expected to have same average energy.

  In this case, ideal MHD results don't carry over to ν, $\eta > 0$.

  Relaxation to ideal state may take a long time.

  Dissipation and $B_o \neq 0$ lead to **anisotropy**: $E(\mathbf{k}_\perp) > E(\mathbf{k}_\|)$;

  also, **anisotropy** persists as k increases.
- Special case: $\mathbf{B}_o \cdot \mathbf{\Omega}_o \neq 0$, $\mathbf{B}_o \times \mathbf{\Omega}_o = 0$; $H_C - (\Omega_o/B_o)H_M$ = ideal invariant.
- Need to incorporate $\mathbf{B}_o$ and $\mathbf{\Omega}_o$ into stat mech & LES; TBD in Session 4.