MHD SGS Turbulence Models Derived from VMS Formulations

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A Survey of VMS Analysis and SGS Models

- Resistive, Incompressible MHD

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left(\mathbf{u} \otimes \mathbf{u} - \frac{1}{\mu_0 \rho} \mathbf{B} \otimes \mathbf{B} \right) + \nabla P - \nu \nabla^2 \mathbf{u} - \mathbf{f} = 0 \quad \nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left(-\mathbf{u} \otimes \mathbf{B} + \mathbf{B} \otimes \mathbf{u} \right) + \nabla r - \lambda \nabla^2 \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

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- Variational Form

$$\mathcal{A}(\mathbf{W}, \mathbf{U}) = (\mathbf{W}, \mathbf{F})$$

$$\mathbf{U} = [\mathbf{u},\ P,\ \mathbf{B},\ r]^T$$

Exact decomposition

$$\mathbf{U} = \mathbf{U}^h + \mathbf{U}'$$

Exact Solution → $\mathbf{U}$; Grid Scales → $\mathbf{U}^h$; Subgrid Scales → $\mathbf{U}'$

Grid scales obtained via a projection operator onto the finite dimensional space.

$$\mathbf{U}^h = \mathbb{P}^h \mathbf{U}$$

Projection Operator → $\mathbb{P}^h$

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$$\mathbf{r}_V\left(\mathbf{U}^h\right) = \frac{\partial \mathbf{u}^h}{\partial t} + \nabla\cdot\left(\mathbf{u}^h\otimes\mathbf{u}^h - \frac{1}{\mu_0\rho}\mathbf{B}^h\otimes\mathbf{B}^h\right) + \nabla P^h - \nu\nabla^2\mathbf{u}^h - \mathbf{f}^h \approx 0 \qquad \nabla\cdot\mathbf{u}^h \approx 0$$

$$\mathbf{r}_I\left(\mathbf{U}^h\right) = \frac{\partial \mathbf{B}^h}{\partial t} + \nabla\cdot\left(-\mathbf{u}^h\otimes\mathbf{B}^h + \mathbf{B}^h\otimes\mathbf{u}^h\right) + \nabla r^h - \lambda\nabla^2\mathbf{B}^h \approx 0 \qquad \nabla\cdot\mathbf{B}^h \approx 0$$

- Variational Form

$$\mathcal{A}\left(\mathbf{W},\mathbf{U}\right) = \left(\mathbf{W},\mathbf{F}\right)$$

$$\mathbf{U} = \left[\mathbf{u},\ P,\ \mathbf{B},\ r\right]^T$$

Exact decomposition

Exact Solution → $\mathbf{U} = \mathbf{U}^h + \mathbf{U}'$ ← Subgrid Scales

($\mathbf{U}^h$ ↑ Grid Scales)

- Leads to residual-based methods

$$\mathbf{U}' \propto \mathcal{R}\left(\mathbf{U}^h\right) \qquad \mathbf{U}^h \to \mathbf{U} \Rightarrow \mathbf{U}' \to \mathbf{0}$$

Residual ⟶ $$\mathcal{R}\left(\mathbf{U}^h\right) = \begin{bmatrix} \mathbf{r}_V\left(\mathbf{U}^h\right) \\ \nabla\cdot\mathbf{u}^h \\ \mathbf{r}_I\left(\mathbf{U}^h\right) \\ \nabla\cdot\mathbf{B}^h \end{bmatrix}$$

As the numerical solution approaches the exact solution the subgrid effects become less significant

Homogeneous, Isotropic Turbulence

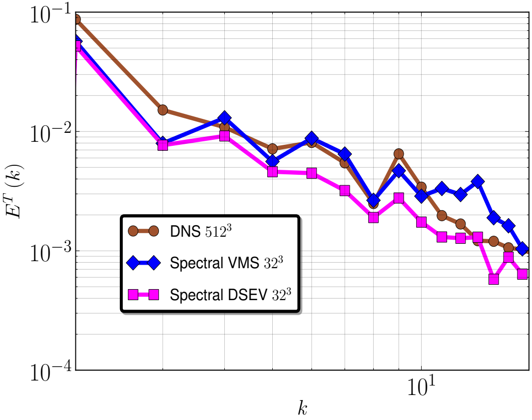
VMS Spectral Method

- Momentum and induction cross correlation terms
- Periodic boundary conditions

  $\Rightarrow$ Only nonlinear terms
- Adds to stability of method

Further Explorations

- Detailed turbulence statistics
- Subgrid dynamo and VMS
- Helical flows
- Vary Pr_m



The VMS-based model performs very well. The dynamic Smagorinsky model is overly dissipative.

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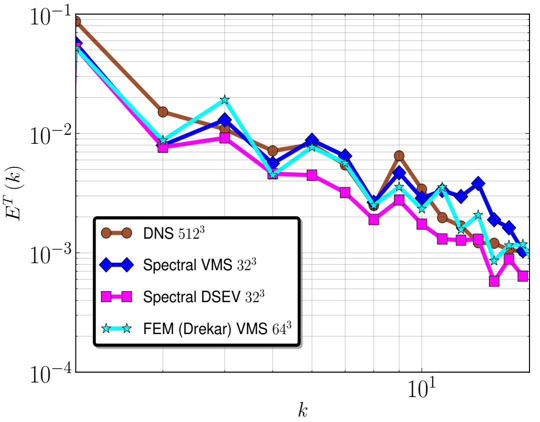

The FEM solution with linear elements using the VMS model also performs very well.

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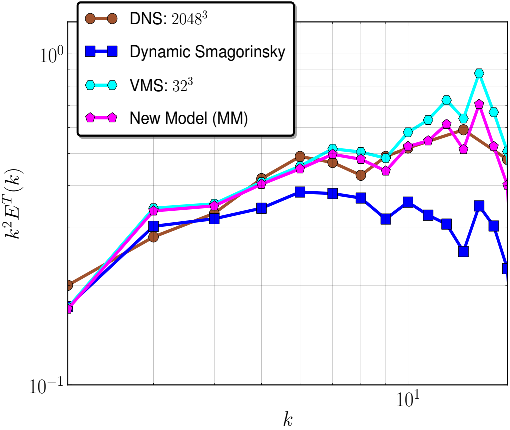


A new, VMS-based mixed model performs exceptionally well for a high Reynolds number flow [a].

[a] DNS data from Pouquet et al. (2010)

Wall-Bounded Turbulence

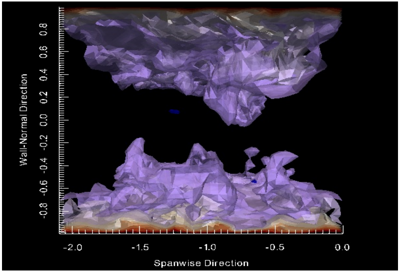
FEM Code Drekar

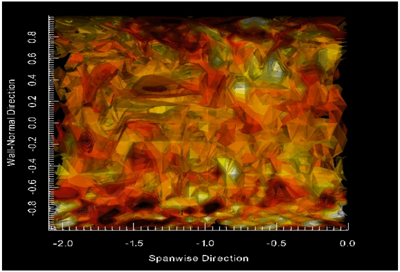
(With: John Shadid, Tom Smith, Eric Cyr, and Roger Pawlowski at SNL)

- Fully implicit Newton-Krylov
- AMG
- Highly scalable: MHD run on 128K cores
- Q1, Q2 elements; edge and face elements
- UQ tools; Adjoint methods for error estimation and sensitivity studies

Channel Flow Challenges and Opportunities

- Pr_m very far from unity ($\sim \mathcal{O}\left(10^{-6}\right)$)
  - Overcome with quasi-static approximation
- DNS of full MHD not available
- LES model convergence studies





Top: Velocity Isosurfaces
Bottom: Induction Isosurfaces

4 Supplementary Slides

Variational Multiscale (VMS) Formulation (I)

Hughes (1995)

- Define *optimal* numerical solution: Optimal projection of $\mathbf{U}$

$$\mathbf{U}^h = \mathbb{P}^h \mathbf{U} \qquad \mathbb{P}^h \rightarrow \text{User-selected projector.}$$

- Subgrid solutions consist of “leftovers”

$$\begin{aligned} \mathbf{U}^{'} &= \mathbf{U} - \mathbf{U}^h \\ &= \mathbf{U} - \mathbb{P}^h \mathbf{U} \\ &= \left(\mathbb{I} - \mathbb{P}^h\right) \mathbf{U} \\ &= \mathbb{P}^{'} \mathbf{U} \qquad \mathbb{P}^{'} \rightarrow \text{Fine scale projector.} \end{aligned}$$

- Solution decomposition

$$\mathbf{U} = \mathbf{U}^h + \mathbf{U}^{'}.$$

Variational Multiscale (VMS) Formulation (II)

- Leads to two problems

$$\mathcal{A}\left(\mathbf{W}^h, \mathbf{U}^h + \mathbf{U}^{'}\right) = \left(\mathbf{W}^h, \mathbf{F}\right) \quad \rightarrow \text{Solve for } \mathbf{U}^h$$
$$\mathcal{A}\left(\mathbf{W}^{'}, \mathbf{U}^h + \mathbf{U}^{'}\right) = \left(\mathbf{W}^{'}, \mathbf{F}\right) \quad \rightarrow \text{Solve for } \mathbf{U}^{'}.$$

- Approximate solution to fine-scale problem (Bazilevs et al. (2007))

$$\mathbf{U}^{'} \approx -\mathbb{P}^{'}\boldsymbol{\tau}\mathbb{P}^{'T}\mathcal{R}\left(\mathbf{U}^h\right) \quad \star$$

- VMS Statement: *Find* $\mathbf{U}^h \in \boldsymbol{\mathcal{V}}^h$ *s.t.*

$$\mathcal{A}\left(\mathbf{W}^h, \mathbf{U}^h + \mathbf{U}^{'}\right) = \left(\mathbf{W}^h, \mathbf{F}\right) \qquad \forall\ \mathbf{W}^h \in \mathbf{V}^h$$

where $\mathbf{U}^{'}$ *is given by* $\star$.

Comments on VMS

- Includes subgrid effects ($\mathbf{U}'$)
- Allows for the possibility of local inverse energy cascade
- Cross stresses ($\mathbf{U}^h \otimes \mathbf{U}'$) well-represented
  (Wang and Oberai (2010))
- Reynolds stresses ($\mathbf{U}' \otimes \mathbf{U}'$) not adequately modeled
  (Wang and Oberai (2010))

  $\Rightarrow$ Eddy viscosity model?

Variational Statement

Variational statement : *Find* $\mathbf{U} \in \boldsymbol{\mathcal{V}}$ *s.t.*

$$\mathcal{A}\left(\mathbf{W},\mathbf{U}\right) = \left(\mathbf{W},\mathbf{F}\right) \qquad \forall\ \mathbf{W} \in \boldsymbol{\mathcal{V}}$$

$$(a,b) = \int_\Omega ab\ d\Omega$$

$$\mathcal{A}\left(\mathbf{W},\mathbf{U}\right) = \left(\mathbf{w}, \rho\frac{\partial \mathbf{u}}{\partial \mathrm{t}}\right) +$$

$$\left(\mathbf{w}, \rho\nabla\cdot\boldsymbol{\mathcal{N}}^{\mathrm{M}}\right) + \left(\mathbf{w},\nabla P\right) + \left(\mathbf{w},\mu\nabla\mathbf{u}\cdot\mathbf{n}\right)_\Gamma - \left(\nabla\mathbf{w},\mu\nabla\mathbf{u}\right) +$$

$$\left(q,\nabla\cdot\mathbf{u}\right) + \left(\mathbf{c},\frac{\partial \mathbf{B}}{\partial \mathrm{t}}\right) +$$

$$\left(\mathbf{c},\nabla\cdot\boldsymbol{\mathcal{N}}^{\mathrm{I}}\right) + \left(\mathbf{c},\nabla r\right) + \left(\mathbf{c},\frac{\eta}{\mu_0}\nabla\mathbf{B}\cdot\mathbf{n}\right)_\Gamma - \left(\nabla\mathbf{c},\frac{\eta}{\mu_0}\nabla\mathbf{B}\right) +$$

$$\left(s,\nabla\cdot\mathbf{B}\right).$$

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