Anisotropic homogeneous turbulence: Rotating, stratified and/or MHD flows

GTP workshop, **NCAR, USA**

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Questions for the ANISO session ?

- Role of the anisotropic substructure (Alfvénic) for predicting power laws ($k^{-3/2}$ IK, $k^{-5/3}$ Kolmogorov, others k^{-2}, steeper ?)
- Relevance of LES and DNS in periodic boxes, v.s. bounded domains
- Relevance of statistical theory, for weak (e.g. wave turbulence) to strong (e.g. EDQNM) triadic interactions (*with strong anisotropy* ?)
- Anisotropy and (FROM !) dynamics in rotating stratified MHD

Rotating stratified equations: governing equations

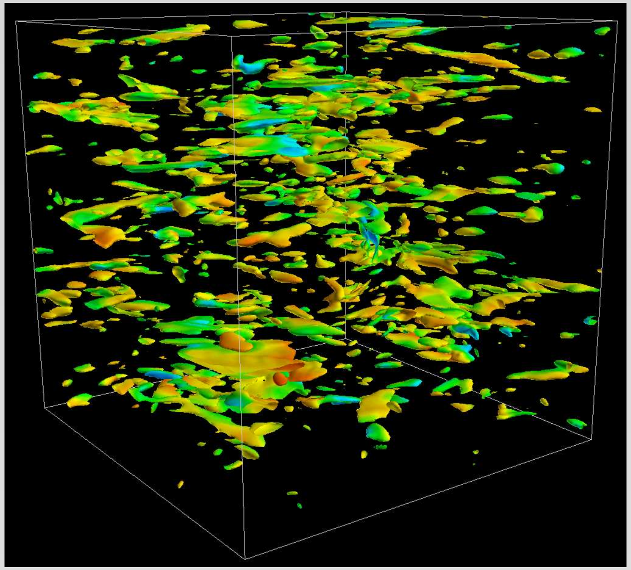
$$\frac{\partial u_i}{\partial t} + \underbrace{f\epsilon_{i3j}u_j}_{\text{Coriolis}} - \underbrace{b\delta_{i3}}_{\text{buoyancy}} + \frac{\partial p}{\partial x_i} = \nu\nabla^2 u_i - u_j\frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial b}{\partial t} + \underbrace{N^2 u_3}_{\text{stratification}} = P_r\nu\nabla^2 b - u_j\frac{\partial b}{\partial x_j}$$

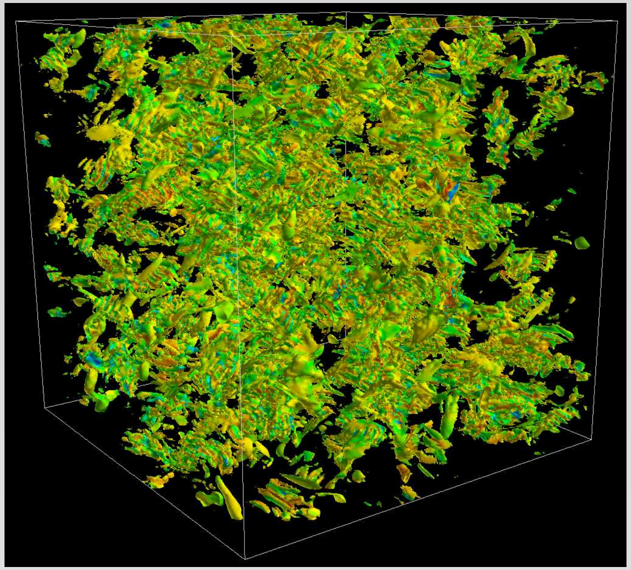
2 external parameters N and f (frequencies)

Valid for a liquid or a gas. P_r characterizes the diffusivity of the stratifying agent (temperature, salt)

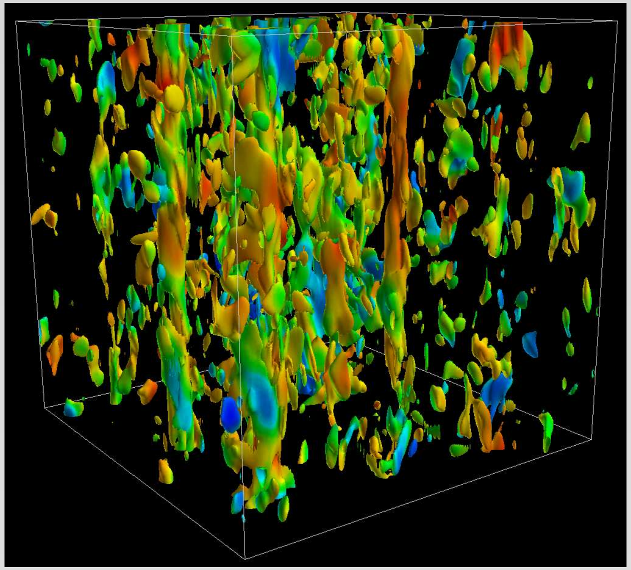
512^3 DNS, Liechtenstein et al. 2005, without mean shear



STRATIFIED



$2\Omega = f = N$



ROTATING

Interest of a toroidal poloidal potential decomp.

- Following the pioneering ‘vortex-wave’ decomposition by Riley *et al.* 1981, toro. polo. dil. in CC & Sagaut book, 2008.
- Connection with solenoidal projection, algebraic in 3D Fourier space, and orthonormal Craya / Herring frame of reference. The buoyancy term is not divergencefree, nor the nonlinear advection term in Navier-Stokes Boussinesq equations: Dil. part to be exactly balanced by Linear and nonlinear parts, with a priori different scalings, of the pressure gradient

A dynamical study, first in Fourier space, at three levels

- Dynamical basic equations in Craya / Herring frame, with buoyancy term scaled as a velocity spectrum
- Generalized Lin's equations for second-order statistics, energies (tor. pol. pot.) and fluxes (poloidal buoyancy flux)
- Equations for third-order statistics, triads, detailed conservation laws and robust closures (EDQNM ...)

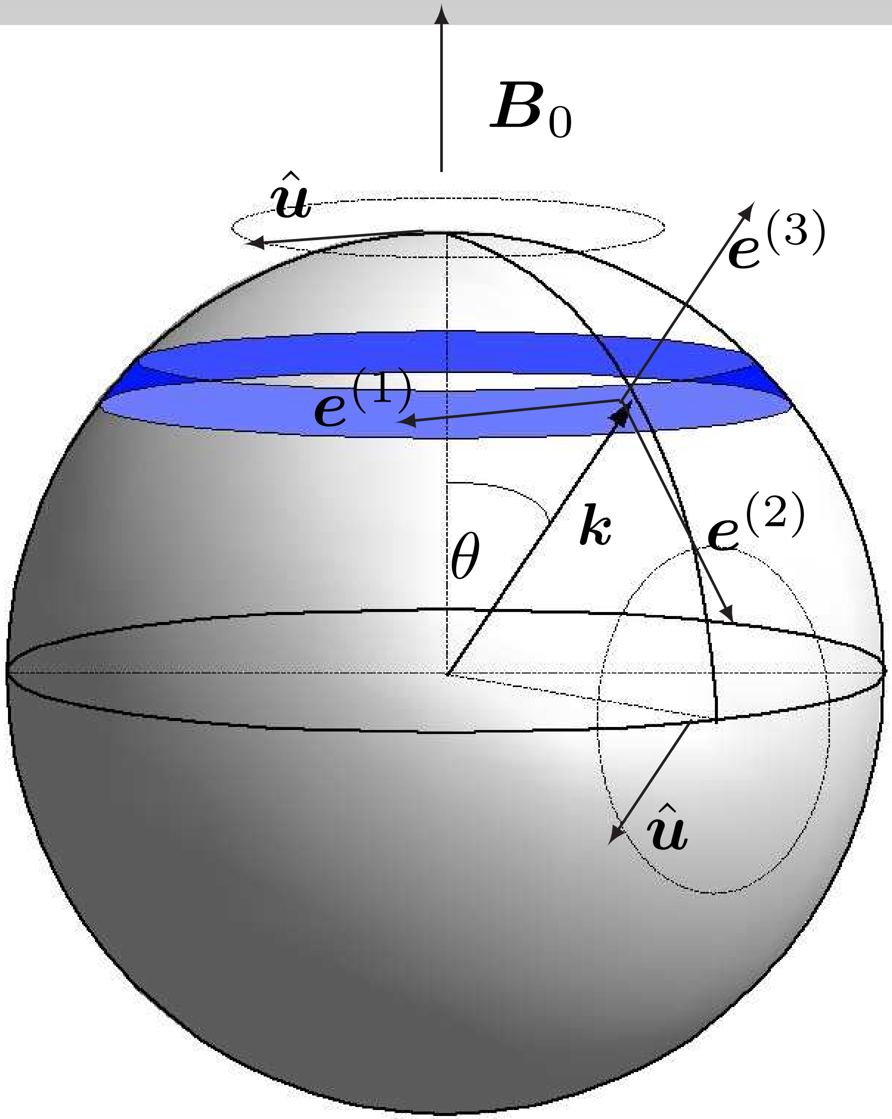


Figure 1: Craya-Herring frame $(\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)})$ in Fourier space.

Stably stratified turbulence, three modes

From N-S Boussinesq equations, solenoidal projection (symbolically), Poisson equation for dilatational balance $\leftarrow \boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$

$$\left(\partial_t - \nu\nabla^2\right)\boldsymbol{u} + (b\boldsymbol{n})^{(sol)} = -\left(\boldsymbol{\omega}\times\boldsymbol{u}\right)^{(sol)}$$

- Equation for the toroidal mode, $\boldsymbol{e}^{(1)}$ in Craya-Herring frame

$$\left(\partial_t + \nu k^2\right)u^{(1)} = \boldsymbol{e}^{(1)}\cdot\widehat{\boldsymbol{\omega}\times\boldsymbol{u}}$$

- Equation for the poloidal mode, $\boldsymbol{e}^{(2)}$

$$\left(\partial_t + \nu k^2\right)u^{(2)} + N\sin\theta_k u^{(3)} = -\boldsymbol{e}^{(2)}\cdot\widehat{\boldsymbol{\omega}\times\boldsymbol{u}}$$

with $\hat{\boldsymbol{u}} = u^{(1)}\boldsymbol{e}^{(1)} + u^{(2)}\boldsymbol{e}^{(2)}$, $\hat{\omega} = \imath k\left(u^{(1)}\boldsymbol{e}^{(2)} - u^{(2)}\boldsymbol{e}^{(1)}\right)$ and equation for $u^{(3)} = -\hat{b}/N$ ... Where is the N-saling (leading Froude-scaling) ??

Generalized Lin's equations

Conservation of the toroidal mode $u^{(1)}$, gravity waves affecting the poloidal mode $u^{(2)}$ and the buoyancy mode $\widehat{b}$, scaled as a pseudo-dilatational velocity component $u^{(3)}$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(tor)}(k, \cos\theta_k, t) = T^{(tor)}(k, \cos\theta_k, t)$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(pol)+(pot)}(k, \cos\theta_k, t) = T^{(W)}(k, \cos\theta_k, t)$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 + 2\imath N \sin\theta_k\right) Z'(k, \cos\theta_k, t) = T^{(z')}(k, \cos\theta_k, t)$$

$$Z' = \frac{1}{2}\left(e^{(pol)} - e^{(pot)}\right) + \imath\langle u^{(2)} u^{(3)}\rangle$$

Strong and *weak (wave)* turbulence face to face

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(tor)}(k, \cos\theta_k, t) =$$

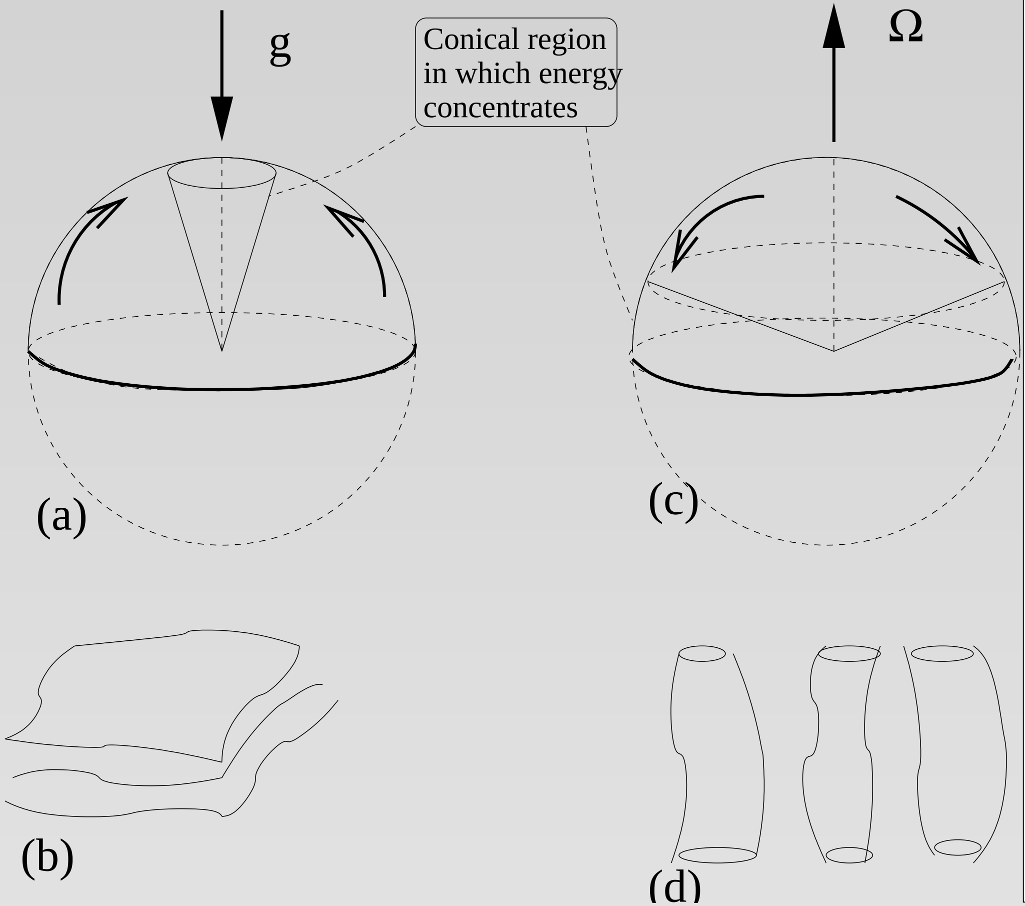
$$\langle u^{(1)*}(\boldsymbol{k},t)\boldsymbol{e}^{(1)}(\boldsymbol{k})\cdot \left(\iiint_{p+q=k} \imath p u^{(1)}(\boldsymbol{p},t)\boldsymbol{e}^{(2)}(\boldsymbol{p}) \times u^{(1)}(\boldsymbol{q},t)\boldsymbol{e}^{(1)}(\boldsymbol{q}) d^3\boldsymbol{p} \right) + \ldots$$

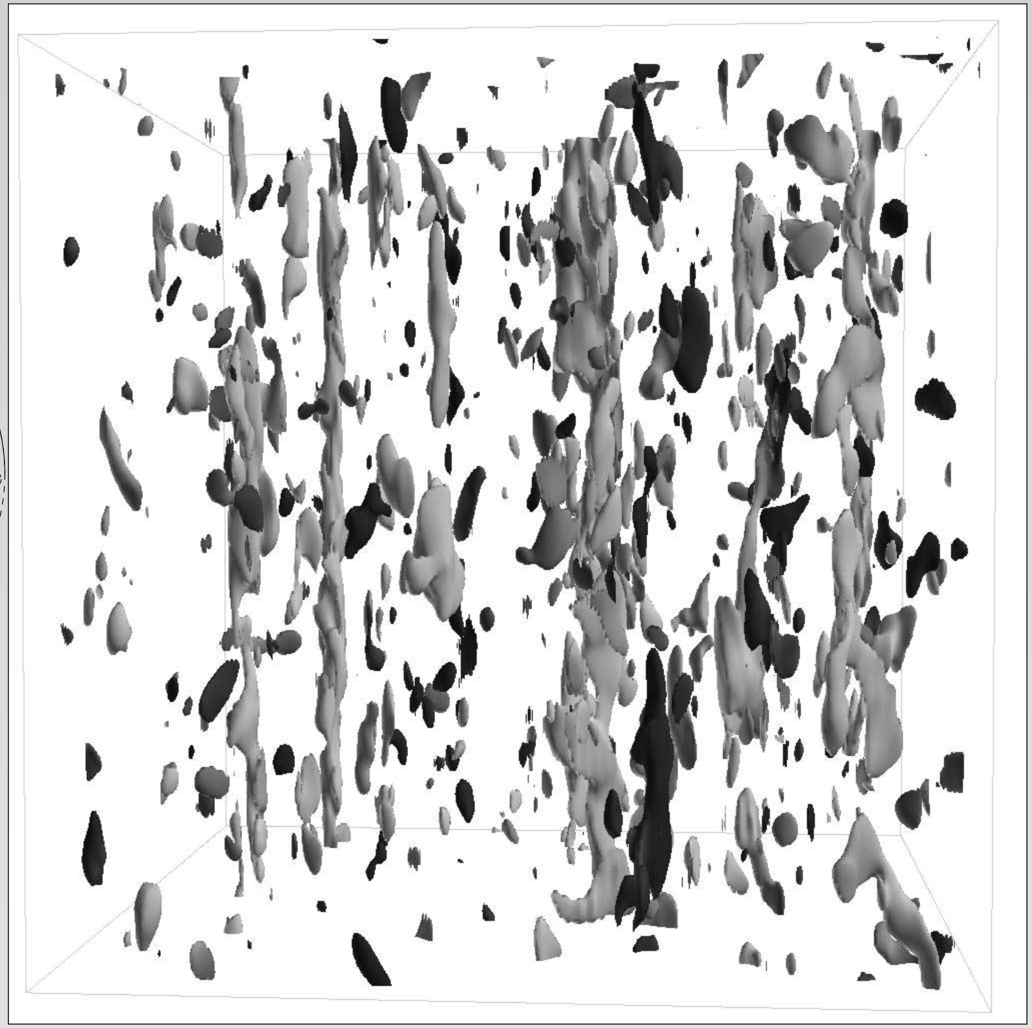
+ other terms in which at least another component than $u^{(1)} = u^{(tor)}$ is present

- The basic eigenmodes decomposition prior to Wave turbulence theory, $\boldsymbol{v} = a_0(\boldsymbol{k},t)\boldsymbol{N}^{(0)} + a_1(\boldsymbol{k},t)\boldsymbol{N}^{(1)}e^{+\imath\sigma_k t} + a_{-1}(\boldsymbol{k},t)\boldsymbol{N}^{(-1)}e^{-\imath\sigma_k t}$, to be reinjected in all equations, with $a_0 = u^{(1)}$, $\sigma_k = N\sin\theta_k$ in SST (extended with rotation), leading to

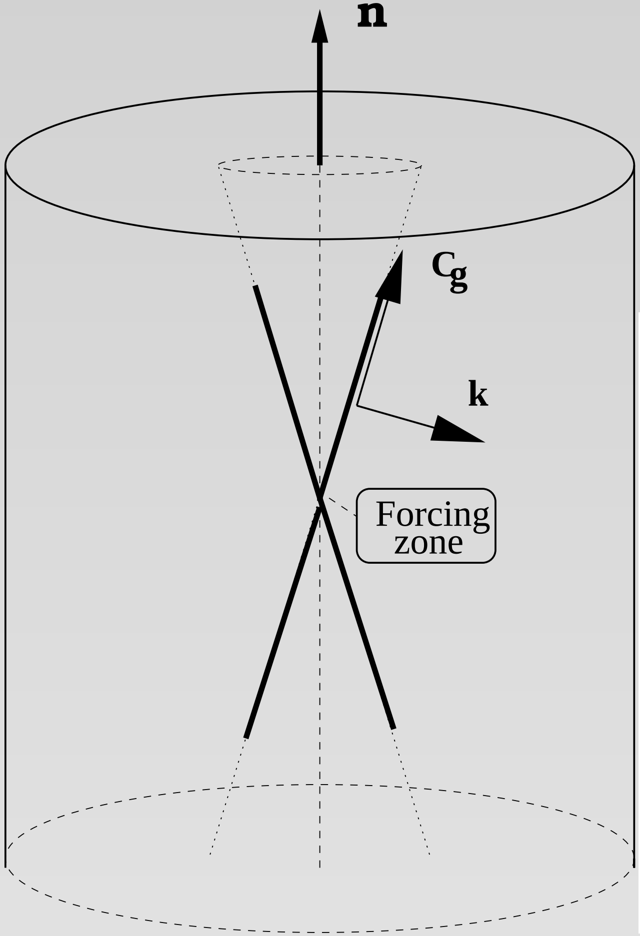
- Disentengling purely toroidal cascade from wave related cascade, as

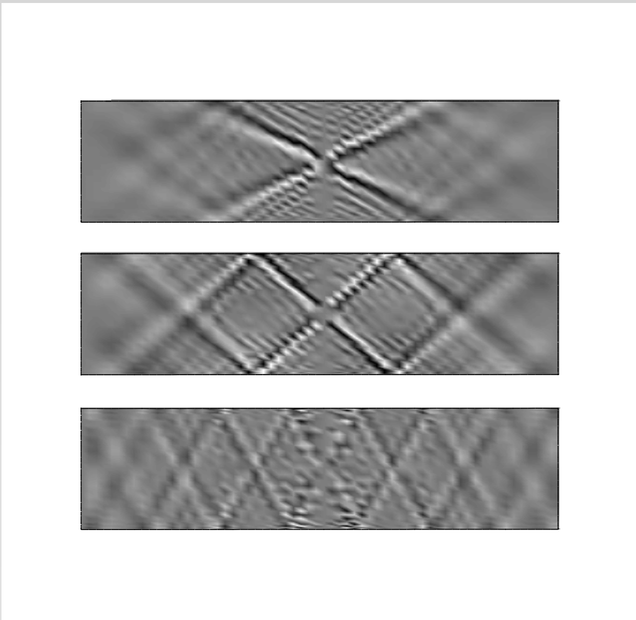
$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(tor)}(k, \cos\theta_k, t) = T^{(tor)}(k, \cos\theta_k, t)_{000} + Fr^{\alpha} T^{(tor)}_{WT}$$


g
Conical region
in which energy
concentrates
Ω
(a)
(c)
(b)
(d)

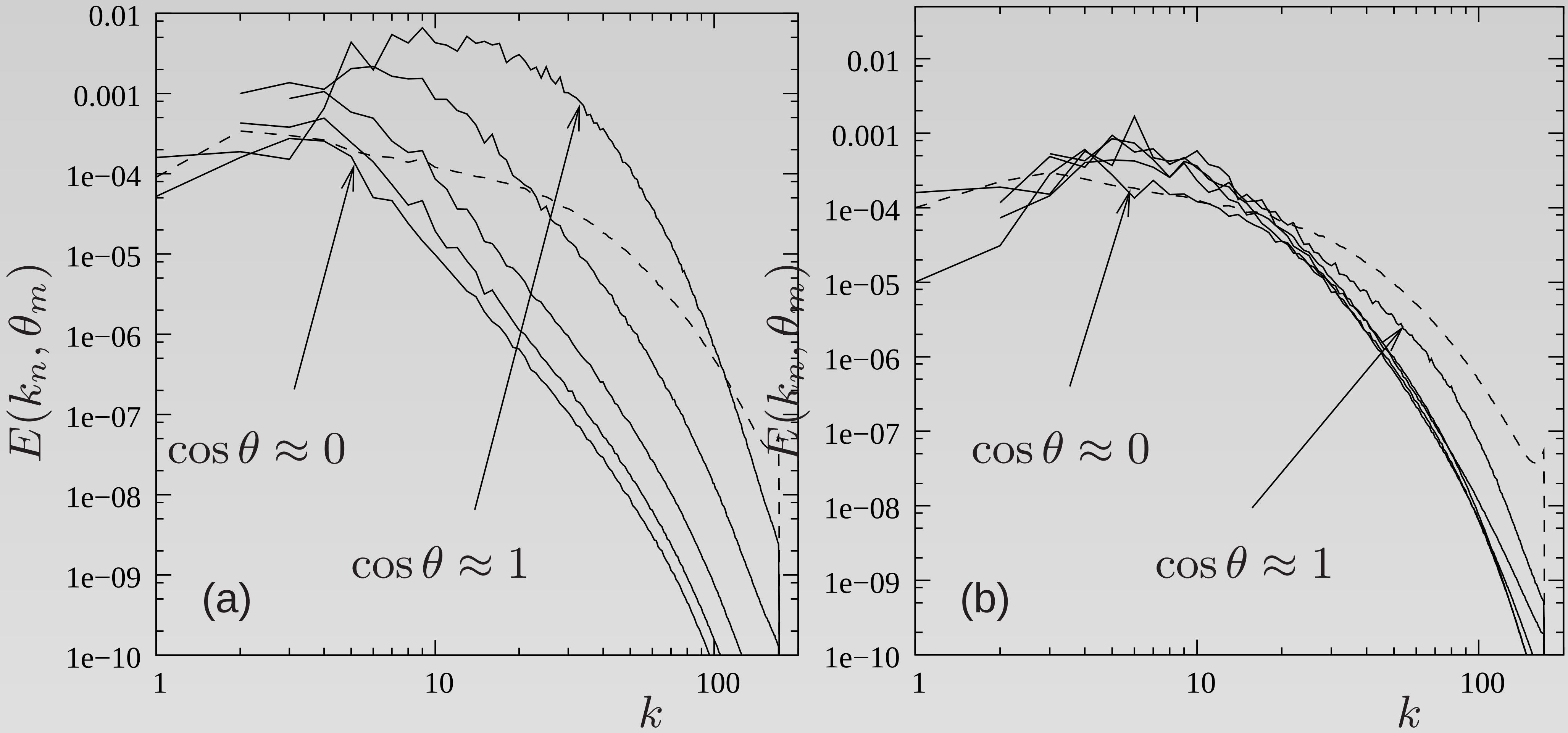


Wave aspects in rotating and/or stratified turbulence

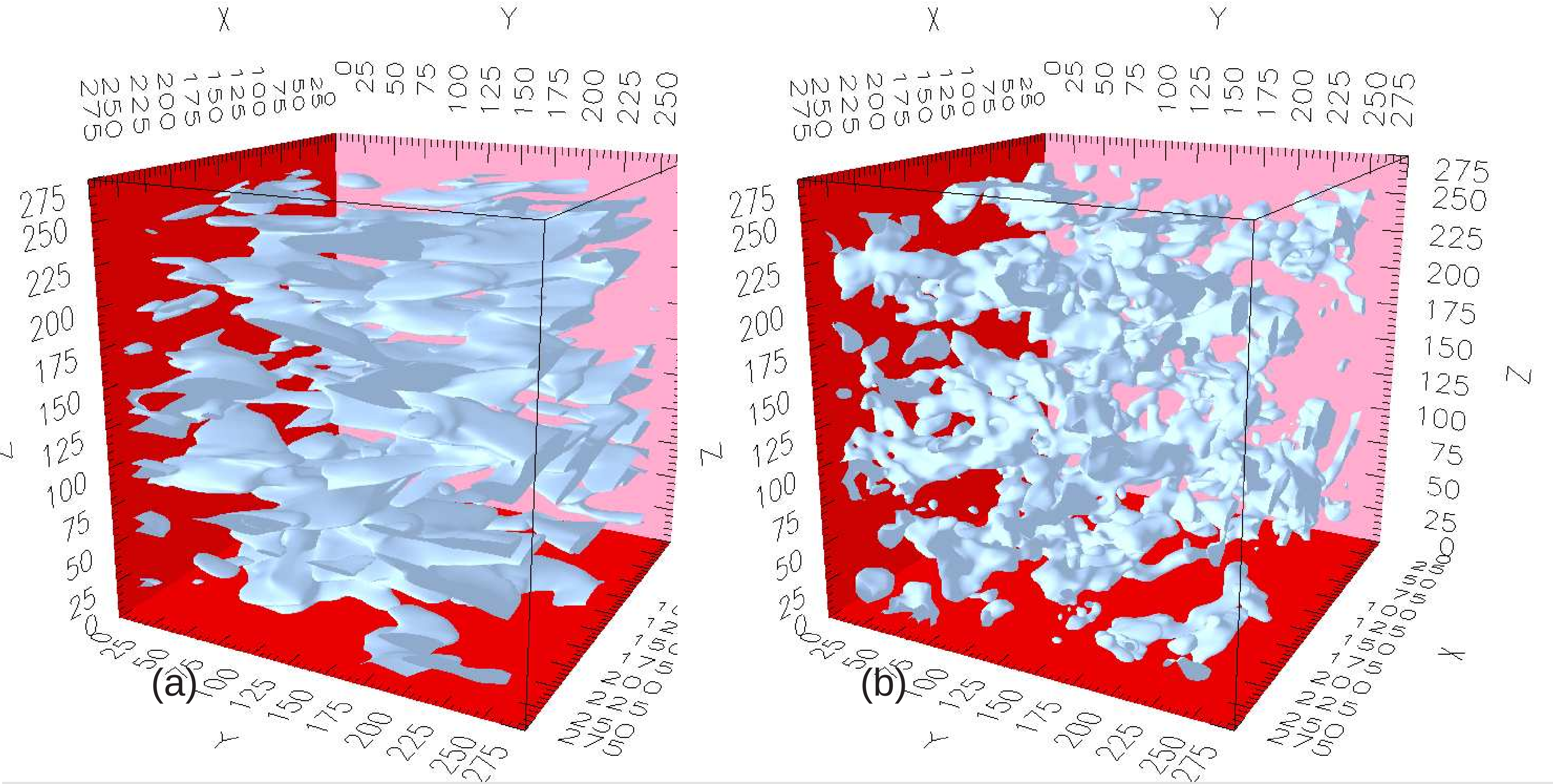




Mc Ewan (1967), Mowbray & Rarity (1967), Godeferd & Lollini, JFM (1999)



Angle-dependent toroidal and poloidal modes (Liechtenstein, 2006)


(a)
(b)
X
Y
Z

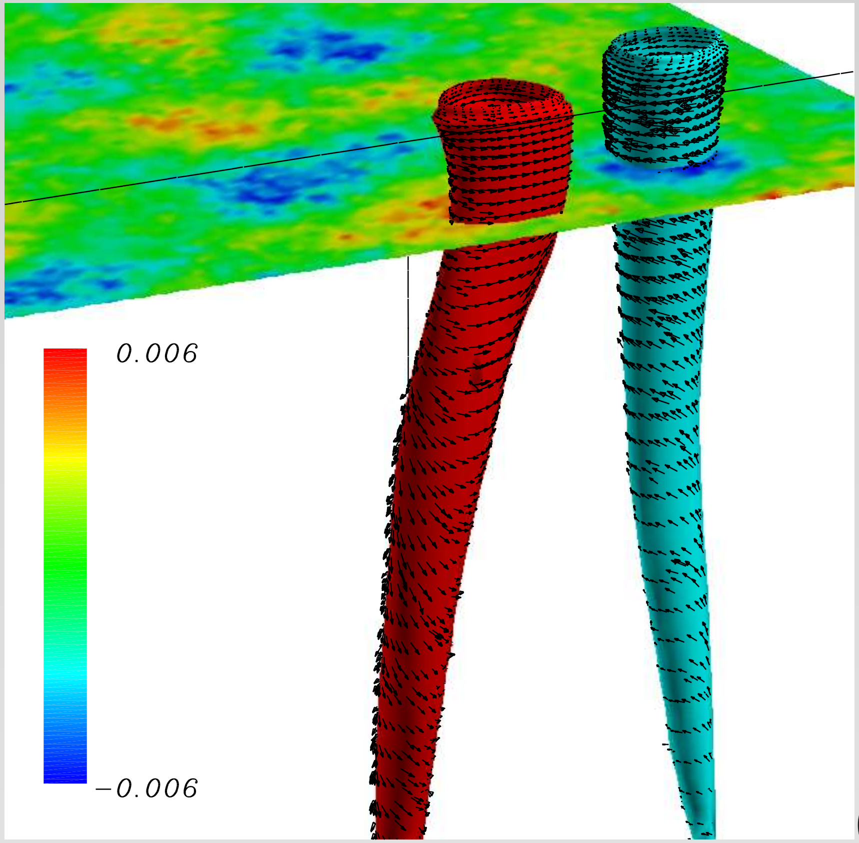
The toroidal cascade and beyond

- The toroidal mode partly decouples from gravity waves. This questions a priori global scalings in terms of Froude number(s): Hanazaki & Hunt (RDT), Lindborg, Chomaz, Billand, Brethouwer, Galtier and coworkers: Vertical Froude number never smaller than 1 (dogma ?!) . Coming back to Riley *et al.* is better, with possibly *small vertical* Froude number

- The toroidal cascade is a 'strong' cascade, vs. a 'weak' gravity-wave turbulence cascade

- It explains the layering (lasagna) even from an initially unstructured state, without need for artificial 2D horizontal forcing. Zig-zag instability suggests only a particular modality of the layering.

What about helicity ?



(Godeferd & Lollini, JFM, 1999)

Helicity nowhere or everywhere ?

- When people understand nothing, helicity is invoked (often wrongly, as in rotating turbulence) ... Beltramization ??
- Local helicity everywhere, statistically relevant helicity spectrum nowhere (Lumley) in homogeneous turbulence ? e.g. artificial forcing (ABC, 50 papers by Mininni, Pouquet etal.) ☹
- Helicity spectrum crucial for the alpha effect, dynamo, etc.
- Revisited (very recently) by Pouquet etal. with stable stratification ☺ with and without rotation

Breaking the mirror symmetry: New relevant statistics

Craya frame ($\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, $\boldsymbol{k}/k$) $\leftrightarrow$ "helical frame" ($\boldsymbol{e}^{(2)} - \imath\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)} + \imath\boldsymbol{e}^{(1)}$, $\boldsymbol{k}/k$)

$$\hat{\mathbf{R}} = \begin{pmatrix} \underbrace{e - \Re Z}_{e^{(tor)}} & \Im Z - \imath\mathcal{H} & F^{(tor)} \\ ImZ + \imath\mathcal{H} & \underbrace{e + \Re Z}_{e^{(pol)}} & F^{(pol)*} \\ F^{(tor)*} & F^{(pol)} & e^{(pot)} \end{pmatrix} = \begin{pmatrix} e - \imath\mathcal{H} & Z & ? \\ Z^* & e + \imath\mathcal{H} & ? \\ ? & ? & ? \end{pmatrix}$$

$\longrightarrow$ Physical space : toro-polo-divergent, helical modes diagonalize the Curl.

Linear coupling for creating net helicity in SST

- Coupling with the toroidal buoyancy flux, with single-point helicity equation by Rorai etal. (PRE, to appear)
  $\frac{dH}{dt} + 2\nu H^{super} + 2N\langle b\omega_{\|}\rangle$ (single-point)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\mathcal{H}(\boldsymbol{k},t) - N\sin\theta_k \Im F^{(tor)} = T^{(H)}$$

- Similarity of Orr -Sommerfeld -Squires variables with toro-polo,
  $\hat{u}_{\|} = -\sin^2\theta_k u^{(2)}$, $\omega_{\|} = \imath k \sin^2\theta_k u^{(1)}$ — $\omega_{\|}$ is also the linearized potential vorticity in SST.

- Other new linear coupling with $\Im Z$ adding solid body rotation.
  $\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\mathcal{H}(\boldsymbol{k},t) = T^{(H)}$ in pure rotation, even with mean shear ☹
  More physical scenarios to create *by linear processes, in strictly homogeneous turbulence* net helicity, without ABC forcing.

Similarly in MHD for velocity and active magnetic vector

Craya frame ($\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, $\boldsymbol{k}/k$) $\leftrightarrow$ "helical frame" ($\boldsymbol{e}^{(2)} - \imath\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)} + \imath\boldsymbol{e}^{(1)}$, $\boldsymbol{k}/k$)

$$\left(\quad \right) = \begin{pmatrix} e - \mathcal{H} & Z & C^1 - C^2 & Z^1 \\ Z^* & e + \mathcal{H} & Z^2 & C^1 + C^2 \\ C^1 - C^2 & Z^2 & e^M - \mathcal{H}^M & Z^M \\ Z^1 & C^1 + C^2 & Z^{M*} & e^M + \mathcal{H}^M \end{pmatrix}$$

Four-component state vector $(u^{(tor)}, u^{(pol)}, b^{(tor)}, b^{(pol)})$ or (u_+, u_-, b_+, b_-), velocity block, magnetic block, cross-block $\langle \hat{\boldsymbol{b}}^* \otimes \hat{\boldsymbol{u}} \rangle$, with C^1 spectrum of cross-helicity and C^2 spectrum of the electromotive force $\boldsymbol{u} \times \boldsymbol{b}$.

Why igoring a complete three-point statistical theory in progress ?

Anisotropic, multimodal EDQNM, possibly matching wave-turbulence theory

- Conventional critiscisms, real drawbacks ?:
  No structures, no (internal) intermittency, specification of the empirical *eddy-damping* term. The anisotropic multimodal aspect, even purely homogeneous, is cumbersome. Questionable infrared behaviour (Davidson ??), wrong FTS (finite time catastrophe) in some cases ?

- Advocating on examples, really solving the generalized Lin equations with EDQNM machinery for generalized three-point (triadic) transfer terms.

- No coherent structures, but anisotropic structuring:
  -) layering in Stably-Stratified Turbulence, saturated one-dimensionalization, via toroidal cascade and angle-dependent toroidal energy spectrum
  -) Saturated two-dimensionalization in rotating turbulence, cyclonic/ anticyclonic preference via third-order vorticity statistics
  -) A two-step scenario of complete two-dimensionalization from 3D isotropic initial data in quasi-static MHD

- Much more refined statistics available (not only energy spectrum and energy transfer) from two-point second-order and three-point third-order: vorticity correlations, structure functions, mixed pressure correlations.

- Much more accurate infrared behavior than in conventional DNS. In addition, purely hydro case, the spectral tensor generated by $\langle \hat{\boldsymbol{u}}^* \otimes \hat{\boldsymbol{u}} \rangle$, of rank 2, is only of rank one in DNS

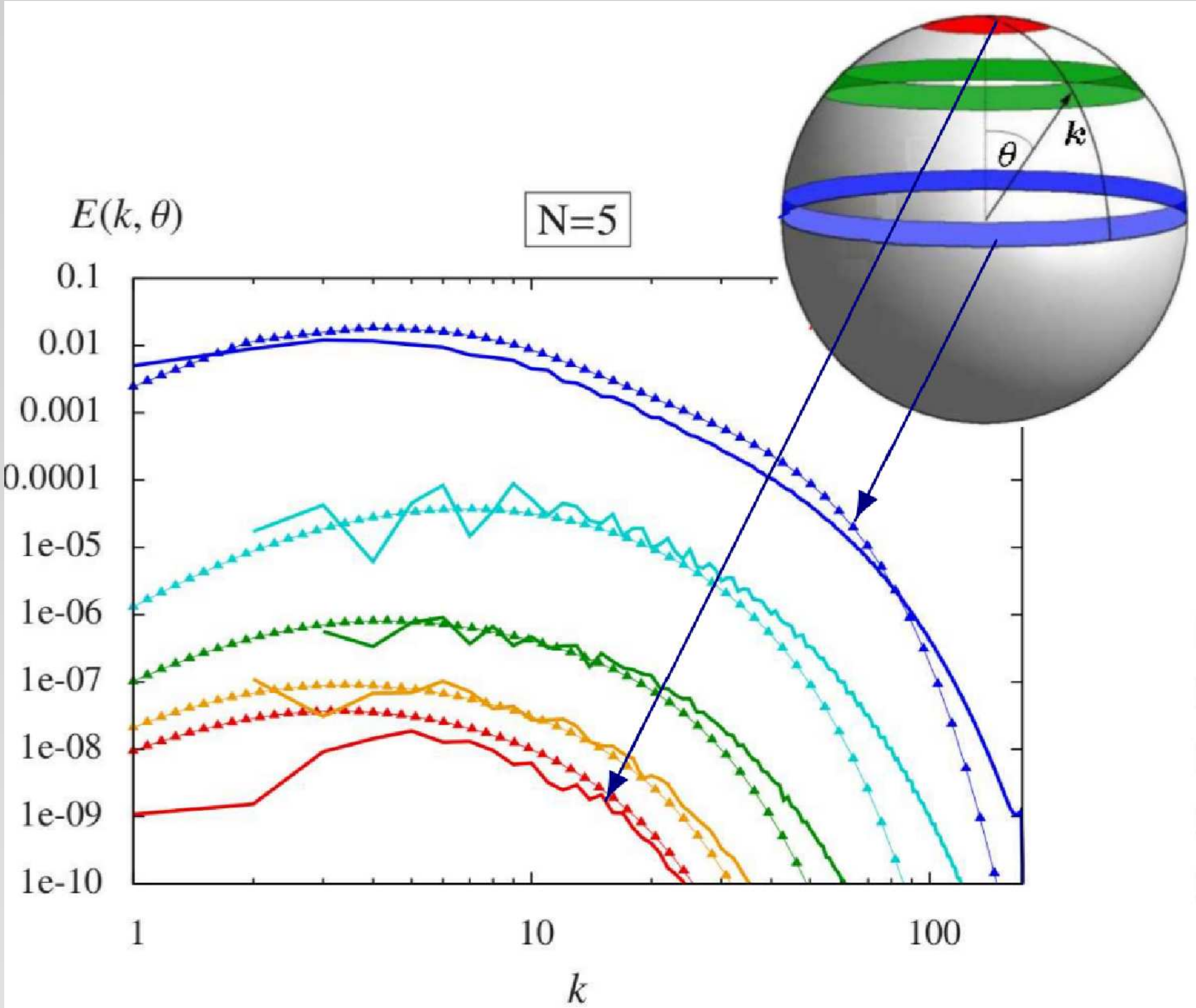
DNS started with a single realization, even with “randomized” phases, and/or with deterministic forcing (e.g. ABC)

$$\langle \hat{\boldsymbol{u}}^* \otimes \hat{\boldsymbol{u}} \rangle \rightarrow s_1^2 = \mathcal{E} + \sqrt{Z^2 + \mathcal{H}^2},\ s_2^2 = \mathcal{E} - \sqrt{Z^2 + \mathcal{H}^2}$$

(the third eigenvalue is zero thanks to divergencefree — algebraic — constraint)

$$\hat{\boldsymbol{u}}^* \otimes \hat{\boldsymbol{u}} \rightarrow s_1^2 = 2\mathcal{E},\ \text{and, because } s_2^2 = 0$$

(e.g. POD, from deterministic — rank one, single nonzero eigenvalue — to weakly random — a limited number of nonzero decreasing eigenvalues —)
Angle dependent directional anisotropy in $\mathcal{E}(\boldsymbol{k}, t)$, polarization anisotropy Z and angle-dependent helicity $\mathcal{H}$ cannot be disentangled in conventional DNS, except by mimicing the ensemble averaging by time or spatial averaging (e.g. averaging on rings in shells in axisym. turbulence, but deterministic ABC is not axisym).



From Favier etal. JFM, 661 (2011),

DNS vs. EDQNM2

Complementary info, details

Pure stratification: Simple existing scalings

- 2D or not 2D ? Charney (1971), Lilly (1983), Lindborg (1999) using third order structure functions from observations.
- Froude numbers, horizontal and vertical, $Fr_h = U/(NL_h)$, $Fr_v = U/(NL_v)$, $L_h \gg L_v$.
- $L_v \sim U/N$ ("zig-zag" instability ? Billant & Chomaz 1999, 2002) $\longrightarrow Fr_v \sim 1$
  Is this *invalidating* previous scalings by Riley *et al.* 1981 ?
- Proposed scaling (Linborg 2006) $E_{hh}(k_h) = C_1 \varepsilon_k^{2/3} k_h^{-5/3}$, $E_p(k_h) \sim C_2 \epsilon_p \epsilon_K^{-1/3} k_h^{-5/3}$, $E_{vv} \sim N^2 k_v^{-3}$.
- $Ri \sim 1/4$ threshold. $Fr^2 Re$ large (Oceanographs, J. Riley) rediscovered (Brethouwer, Linborg, Billant, Chomaz.)

‘Exact’ equations for a single triad. Why not 2D ?

- ‘Pure’ 2D turbulence, the toroidal mode corresponds to the 2D vortical mode at $k_{\|} = 0$. The poloidal mode corresponds to the 2D ‘jettal’ mode at $k_{\|} = 0$
- Detailed conservation of toroidal energy $\partial_t u^{(1)} = \boldsymbol{e}^{(1)} \cdot (\widehat{\omega \times \boldsymbol{u}})$. Contributions other than $u^{(1)}(\boldsymbol{p}, t)$, $u^{(1)}(\boldsymbol{q}, t)$ in the convolution product are affected by wave turbulence phase terms, which are strongly damped in average by phase-mixing, so a toroidal cascade can emerge

$$\dot{u}^{(1)}(\boldsymbol{k}) = G'_{kpq}(p_\perp^2 - q_\perp^2) u^{(1)*}(\boldsymbol{p}) u^{(1)*}(\boldsymbol{q})$$

$$\dot{u}^{(1)}(\boldsymbol{p}) = G'_{kpq}(q_\perp^2 - k_\perp^2) u^{(1)*}(\boldsymbol{q}) u^{(1)*}(\boldsymbol{k})$$

$$\dot{u}^{(1)}(\boldsymbol{q}) = G'_{kpq}(k_\perp^2 - p_\perp^2) u^{(1)*}(\boldsymbol{k}) u^{(1)*}(\boldsymbol{p})$$

The Euler stability for a ... SOLID !

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \qquad \times\Omega_1 \qquad \times I_1\Omega_1$$

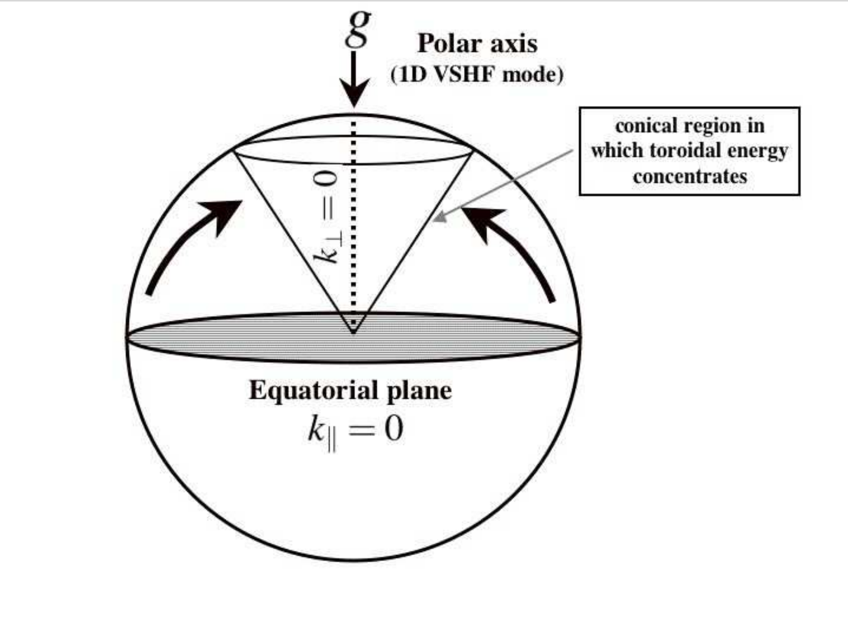
$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1, \qquad \times\Omega_2 \qquad \times I_2\Omega_2$$

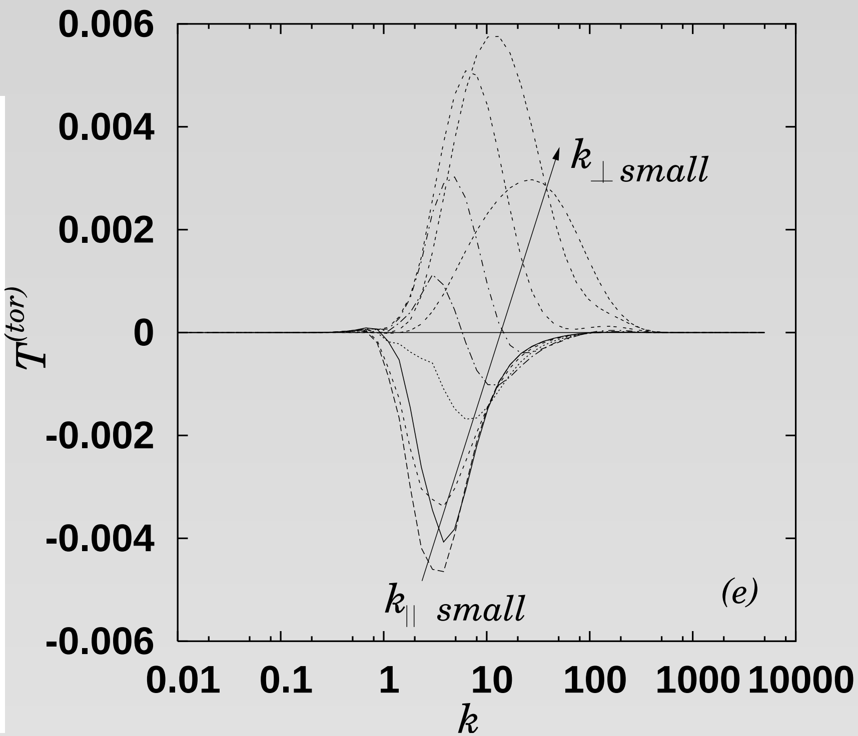
$$I_1\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2, \qquad \times\Omega_1 \qquad \times I_1\Omega_1$$

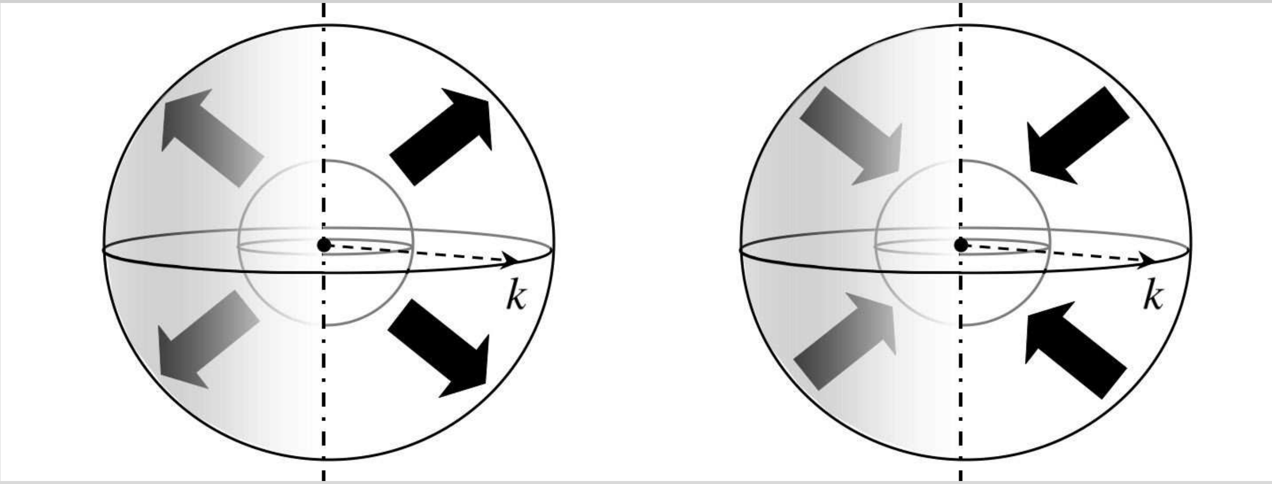
Both conservation of $(1/2)(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$ (kinetic energy) and $(I_1\Omega_1)^2 + (I_2\Omega_2)^2 + (I_3\Omega_3)^2$ (norm of the angular momentum)

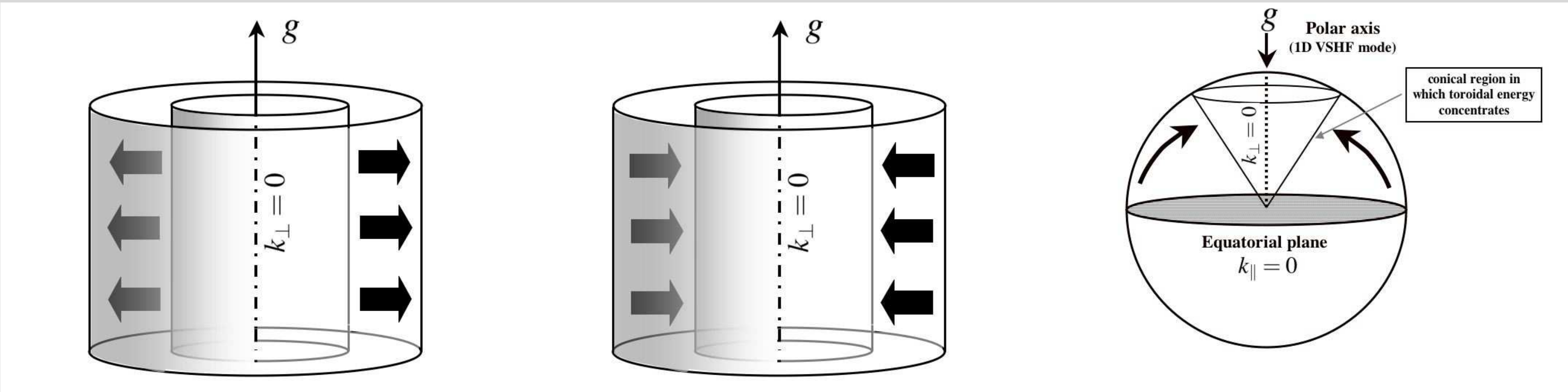
Instability of the motion with respect to the *intermediate* inertia momentum, say I_2

Toroidal main energy drain, ‘wave-released’ EDQNM2






k
k


g
$k_\perp = 0$
g
$k_\perp = 0$
g
Polar axis
(1D VSHF mode)
conical region in which toroidal energy concentrates
$k_\perp = 0$
Equatorial plane
$k_\parallel = 0$

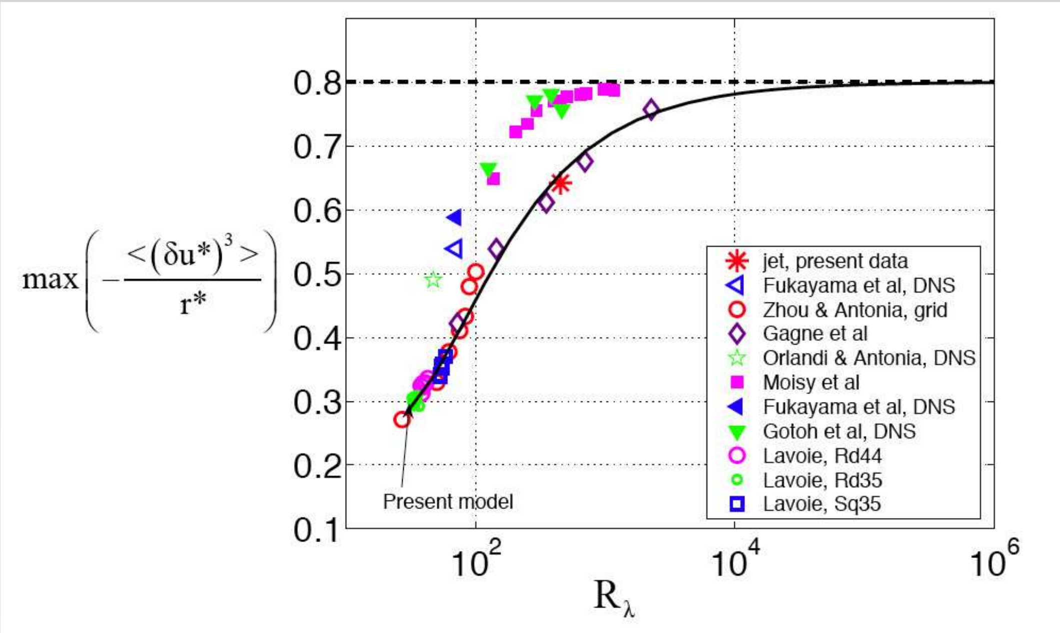
From spectral transfer to third-order structure function

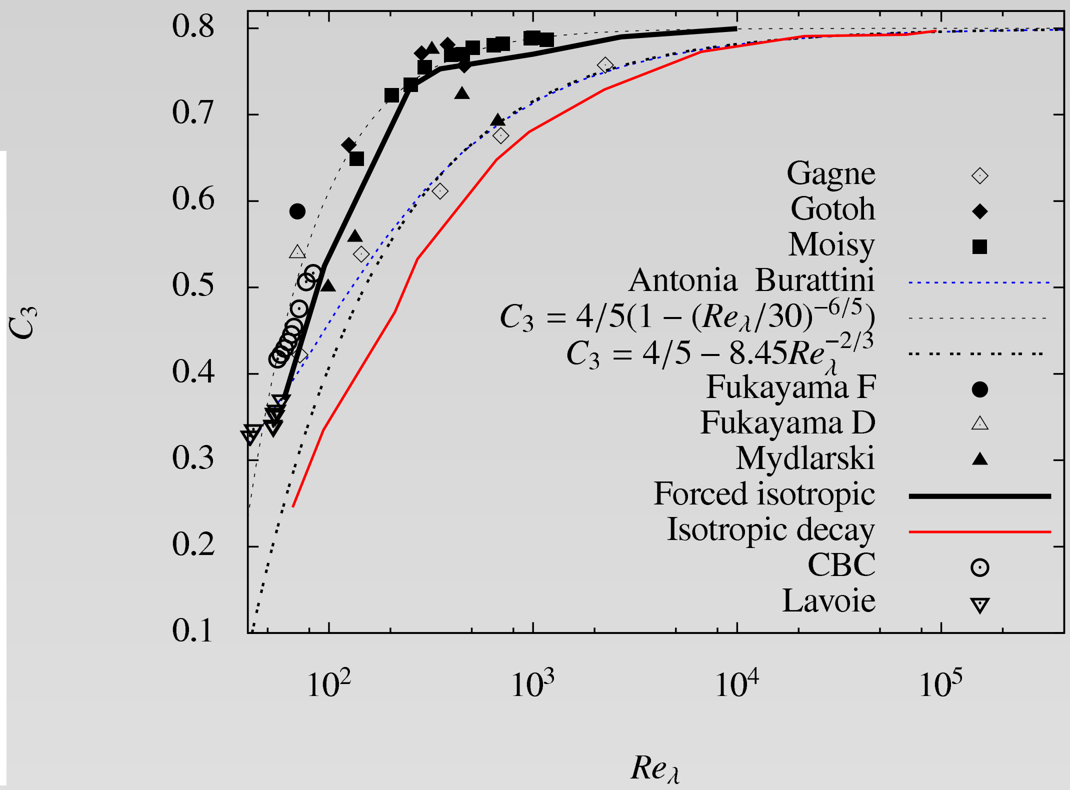
- All second-order (single- and two-point) statistics is derived from $E(k,t)$ (HIT)
- All third-order (single and two-point) statistics is derived from T(k,t)

$$\langle(\delta u_L)^3\rangle = 12r\int_0^\infty g_5(kr)T(k)dk$$

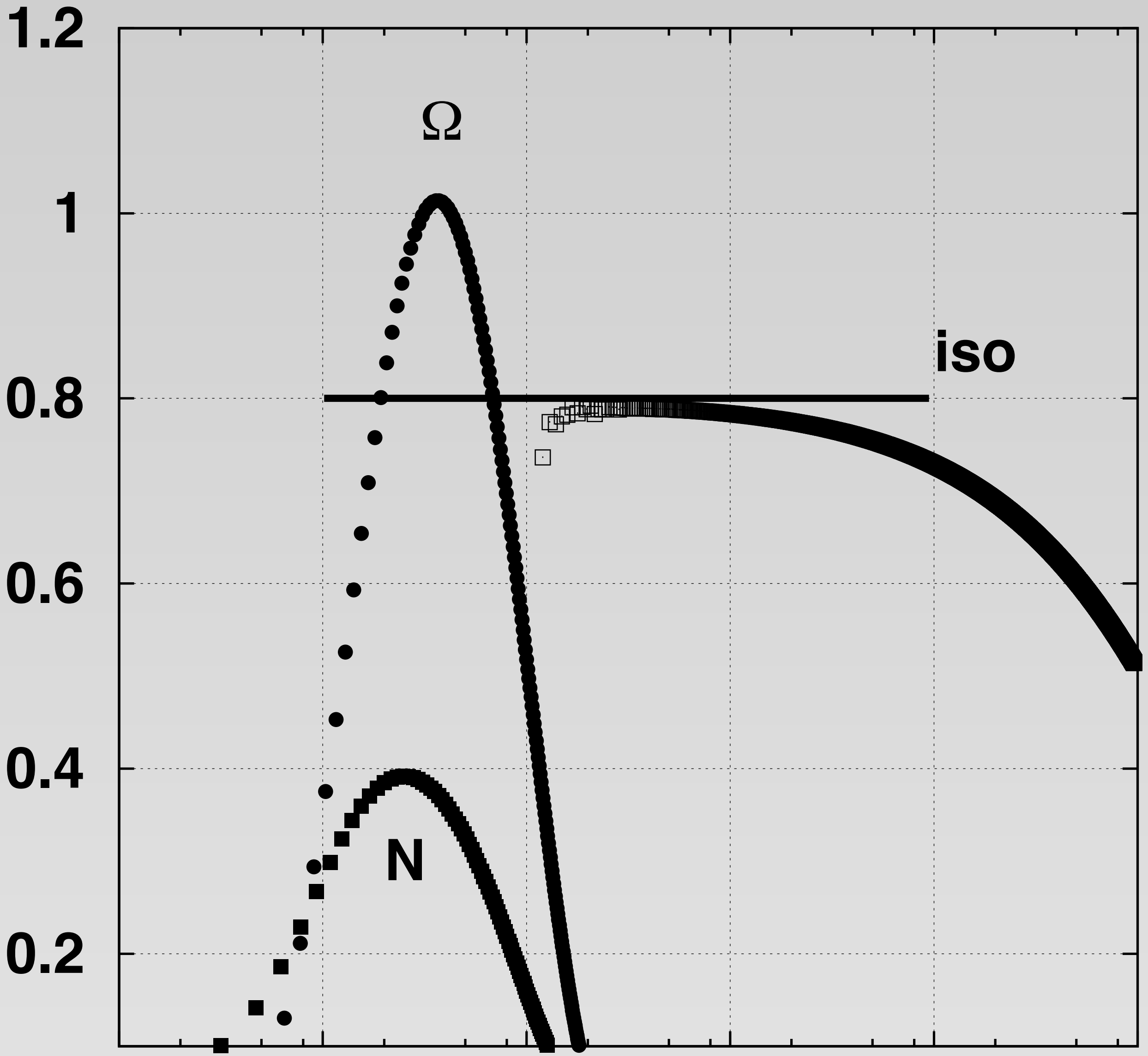
$$g_5(kr) = \frac{3(\sin kr - kr\cos kr) - (kr)^2\sin kr}{(kr)^5} \rightarrow \frac{1}{15} - \frac{1}{210}(kr)^2$$

Revisiting the departure from the ‘exact’ 4/5 Kolmogorov’s law





Antonia & Burratini (2006), Antonia in Cargese 2007, Tchoufag *et al.* PoF submitted.

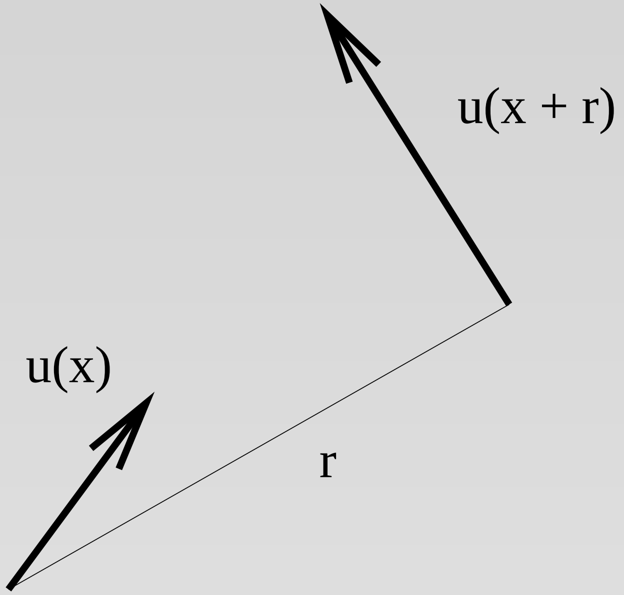


Normalized 4/5 law for $-\langle(\delta u)^3\rangle(r)/(\varepsilon r)$ (F. S. godeferd, Phys. D, to appear)

- The scheme injection - cascade - dissipation can be questionned, for various modalities of energy injection
- $\cdots$ especially in the absence of energy injection ! Rotating turbulence, stably-stratified turbulence (for TOTAL energy), Alfvénic MHD $\cdots$
- Be careful with dimensional analysis: S (spherical strain rate, anisotropic plane strain, plane shear), Ω (rotation rate), N (B-W frequency) can yield very different scalings in ANISOTROPIC turbulence
- Solenoidal projection: Fourier space is not only for computers using DNS $\cdots$
- To avoid a schyzophrenic viewpoint: Systematically compare results and even theory in both space
- Strong relevance of multimodal axisym. EDQNM, to be used now to extract more and more cubic (and fourth order ?) statistics in physical space, third-order structure functions, $\cdots$ etc.

Two-point correlations, or structure functions

- Second-order two-point correlations : $R_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x} + \boldsymbol{r}) \rangle$



-) Single-point, Reynolds Stress tensor, deviator b_{ij}, "componentality"

-) Two-point, directional dependence on $\boldsymbol{r}$, dimensionality

- Similar information in $R_{ij}(\boldsymbol{r})$ and in $\hat{R}_{ij}(\boldsymbol{k})$?

Physical/ Fourier space analogies for anisotropy

- "True" anisotropic scalar (Scalar Spherical Harmonics)

$$\frac{1}{2}R_{ii}(\boldsymbol{r}) \quad \leftrightarrow \quad \frac{1}{2}\hat{R}_{ii}(\boldsymbol{k}) = e(\boldsymbol{k})$$

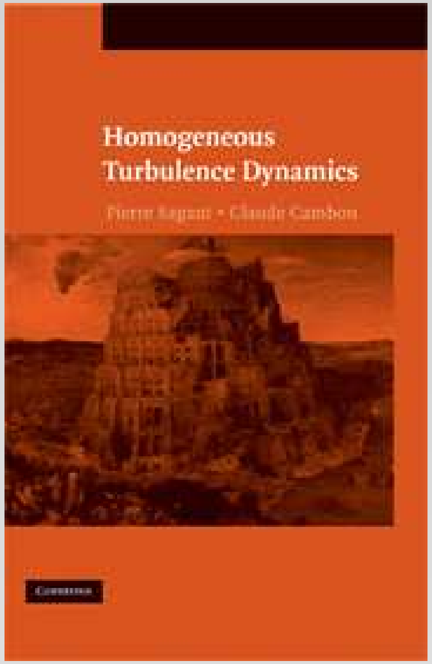
$$\sum r_n^m(r)Y_n^m(\theta_r,\phi_r) \quad \leftrightarrow \quad \sum \varphi_n^m(k)Y_n^m(\theta_k,\phi_k)$$

Same degree, simple integral for swaping from $r_n^m(r)$ to $\varphi_n^{m'}(k)$ (CC & Teissèdre 1985, CRAS Paris) $\longrightarrow$ JFM Aniso (SO(3) symmetry group ?)

- Same expansions in iso, semi-iso, axisym, semi-axisym (isometry)

$$R_{ij} = A\delta_{ij} + B\frac{r_i r_j}{r^2} + Cn_i\frac{r_j}{r} + Dn_i n_j + E\epsilon_{ijn}\frac{r_n}{r} + \cdots$$

$$\hat{R}_{ij} = A'\delta_{ij} + B'\frac{k_i k_j}{k^2} + C'n_i\frac{k_j}{k} + D'n_i n_j + E'\epsilon_{ijn}\frac{k_n}{k} + \cdots$$



Free advertising ! relevance of Homogeneous Anisotropic Turbulence (Homogeneous Turbulence Dynamics, Sagaut & CC, CUP, New York, 2008.