Relaxing ideal magneto-fluids: Eulerian vs semi-Lagrangian approaches

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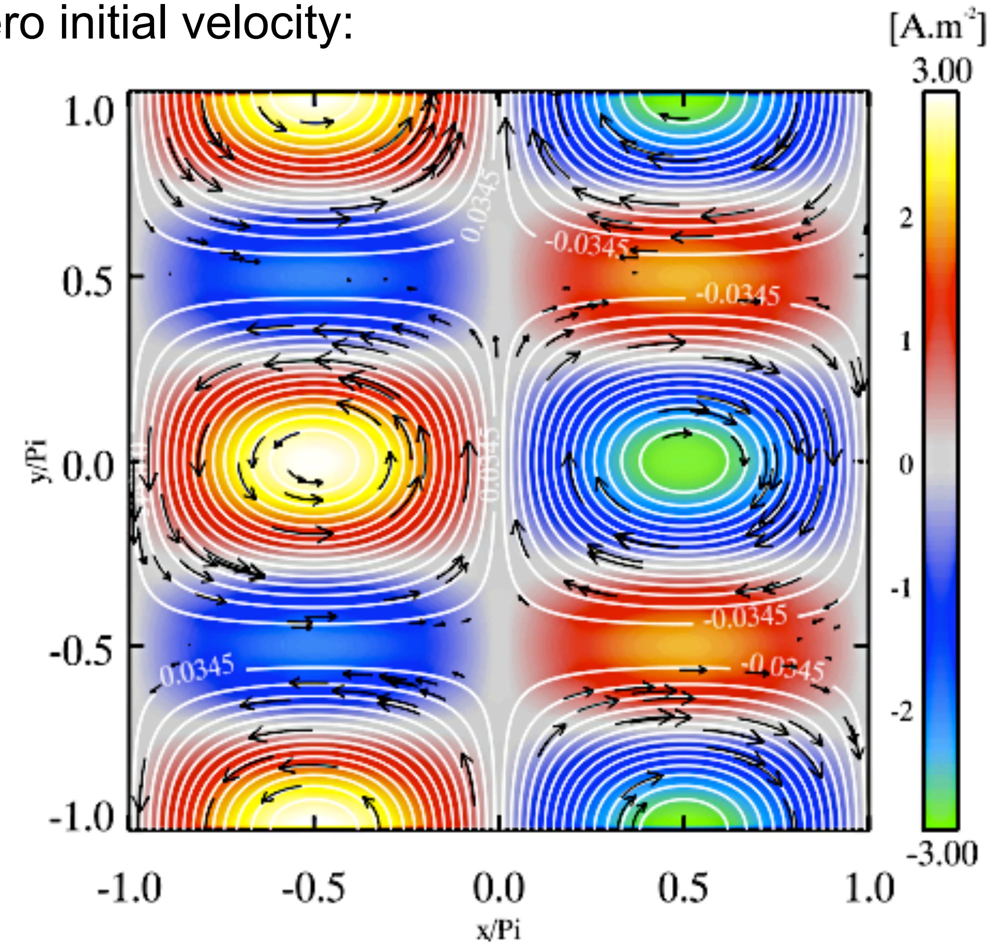
Consider the classical ideal MHD incompressible Navier-Stokes equations with an initial non force-free magnetic field consisting of a magnetic island configuration with zero initial velocity:

$$\frac{d\mathbf{v}}{dt} = -\nabla\pi' + \frac{1}{\rho_o\mu_o}\mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{v} \qquad (34)$$

$$\frac{d\mathbf{B}}{dt} = \mathbf{B}\cdot\nabla\mathbf{v}, \qquad (35)$$

$$\nabla\cdot\rho_o\mathbf{v} = 0\ , \qquad (36)$$

$$\nabla\cdot\mathbf{B} = 0\ , \qquad (37)$$



(34)-(37) can be written compactly as

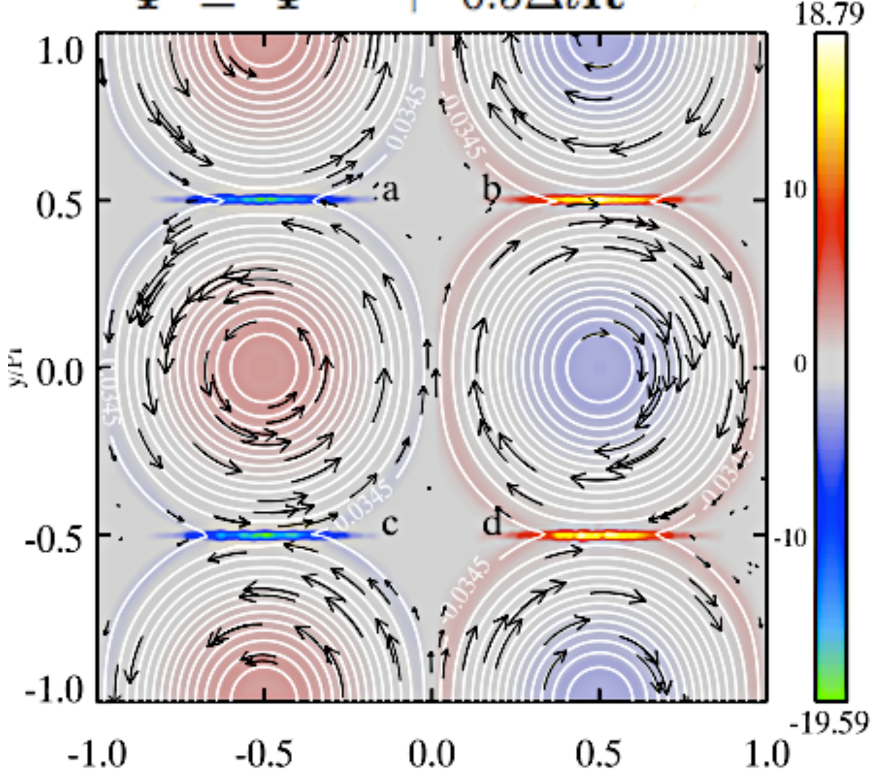
$$\frac{d\,\mathbf{\Psi}}{d\,t} = \mathbf{R} \qquad \mathbf{\Psi} = \{\mathbf{v},\mathbf{B}\}^T \text{ and } \mathbf{R} = \{\mathbf{R_v},\mathbf{R_B}\}^T$$

We integrate (34)-(37) using the non-oscillatory forward-in-time approach of the EULAG-MHD model.

$$\mathbf{\Psi}_\mathrm{i}^n = LE_\mathrm{i}(\widetilde{\mathbf{\Psi}}) + 0.5\Delta t\mathbf{R}_\mathrm{i}^n \ \equiv\ \widehat{\mathbf{\Psi}}_\mathrm{i} + 0.5\Delta t\mathbf{R}_\mathrm{i}^n$$

$$\widetilde{\mathbf{\Psi}} \ \equiv\ \mathbf{\Psi}^{n-1} + 0.5\Delta t\mathbf{R}^{n-1}$$

LE_i : flux-form **Eulerian** / advective **semi-Lagrangian** (SL) two-time level transport operator.



Spontaneous formation of current sheets (tangential discontinuities) at the compression regions.

Analytically: (34)-(37) forbid reconnection.

However: Truncation terms in *LE* represent departures from (34)-(37) and allow for reconnection and change in magnetic topology.

Q: Eulerian and SL advection operators have different dissipative properties. How does this affect the relaxation of the magneto-fluid into a static equilibrium state where

$$-\nabla\pi' + \nabla\cdot(\mathbf{BB})/(\mu_0\rho_0) = 0 \quad \mathbf{?}$$

-SL discretization schemes arise from the path integration

$$\psi(\mathbf{x},t) = \psi(\mathbf{x}_0,t_0) + \int_T R\,dt \quad \text{of the Lagrangian evolution equation} \quad \frac{d\psi}{dt} = R$$

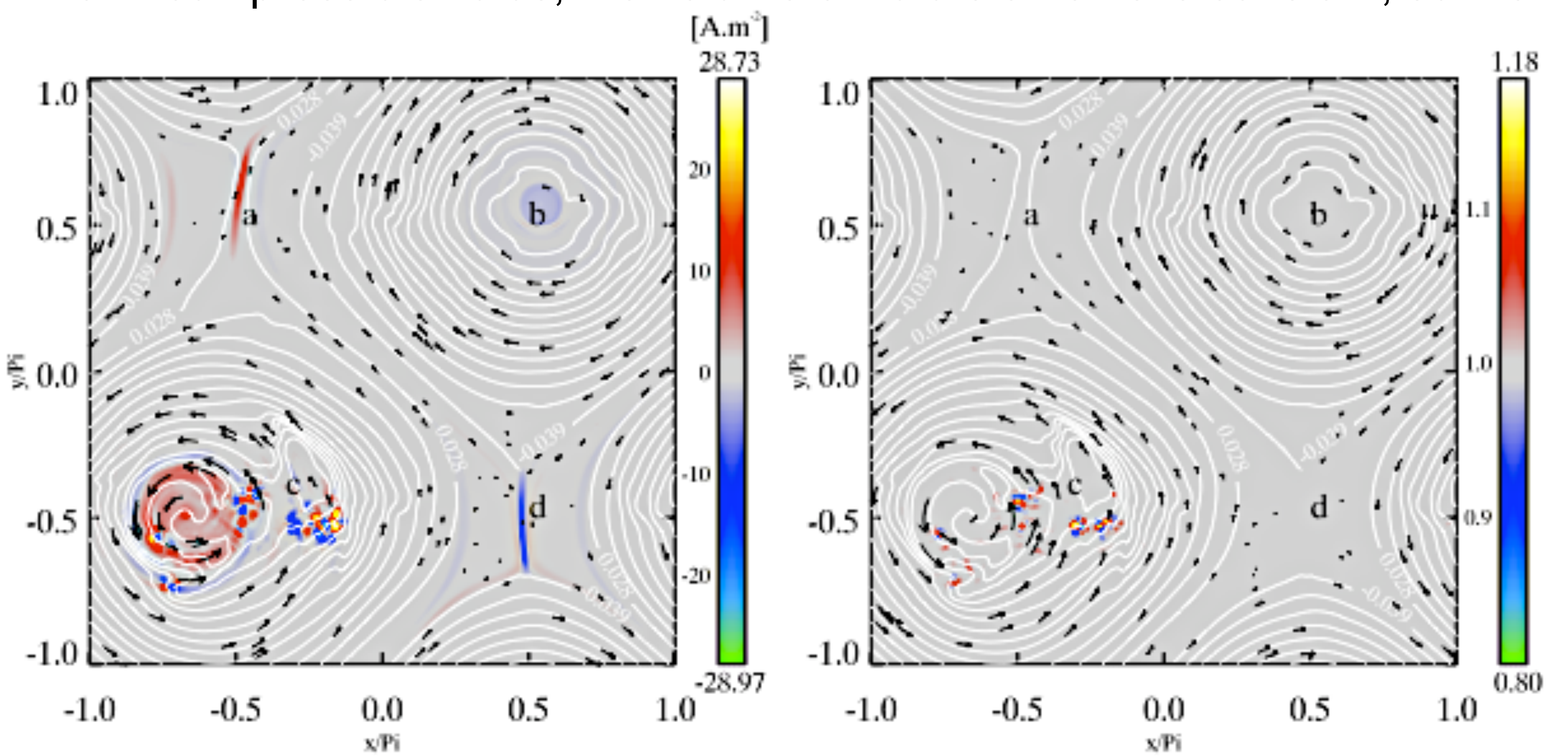
where $\mathbf{x}_0 = \mathbf{x}_\mathrm{i} - \int_{t_0}^{t} \mathbf{v}(\mathbf{x}(\tau),\tau)\,d\tau$

-Especially relevant to the SL approach is the interpretation of flows in terms of a space-time continuum, where the volume of fluid elements evolves in accordance with the Euler expansion formula:

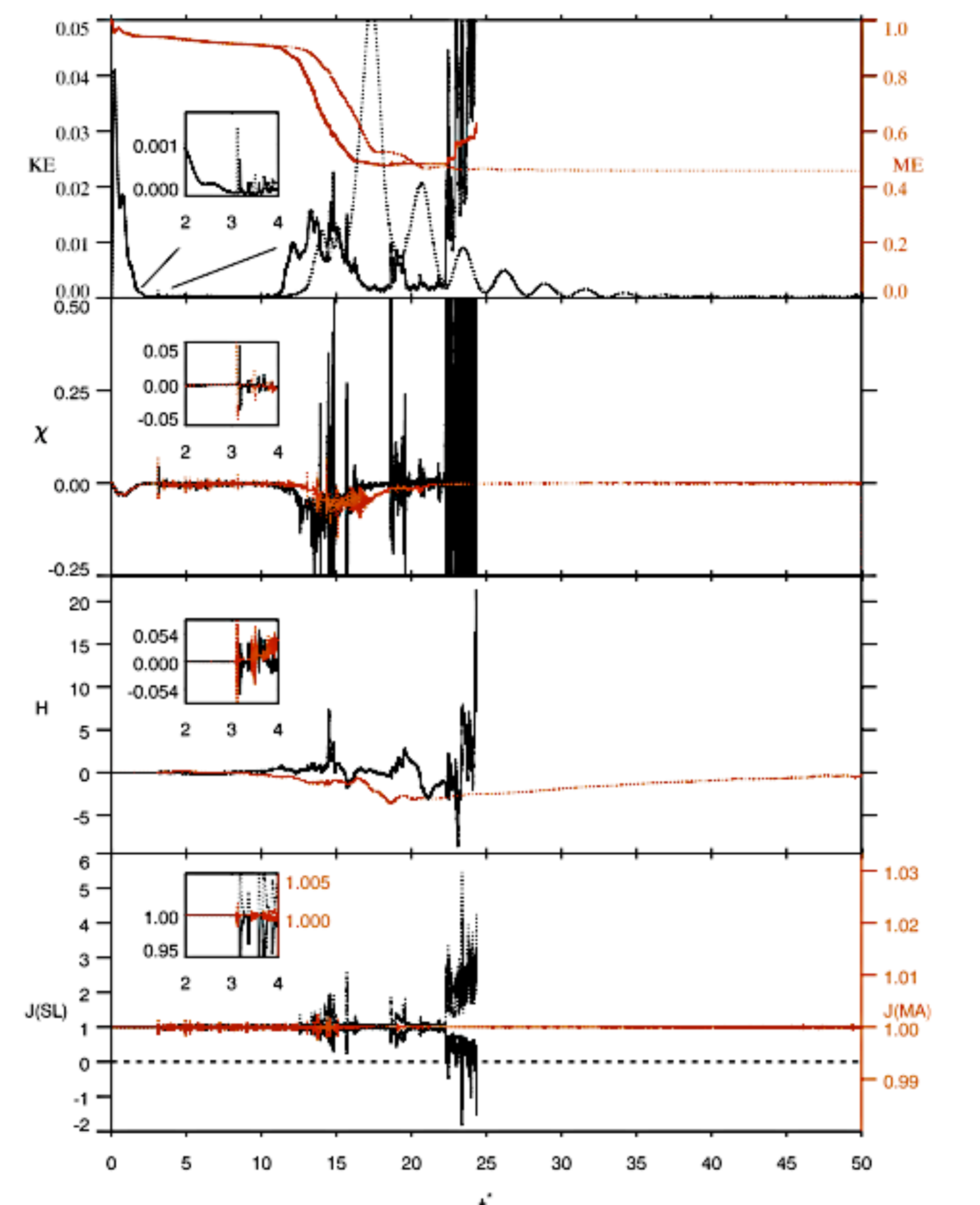
$$\frac{d\ln \mathrm{J}}{dt} = \nabla\cdot\mathbf{v} \qquad \mathrm{J} := \det(\partial\mathbf{x}/\partial\mathbf{x}_0)$$

-Together with the mass continuity equatior $\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v}$ this leads to $\rho(\mathbf{x}_\mathrm{i},t) = \hat{\mathrm{J}}\rho(\mathbf{x}_0,t_0)$.

-For incompressible fluids, the volume of fluid elements is constant, so the inverse flow Jacobian $\hat{\mathrm{J}} \equiv \mathrm{J}^{-1} = \det(\partial\mathbf{x}_0/\partial\mathbf{x}) = 1$



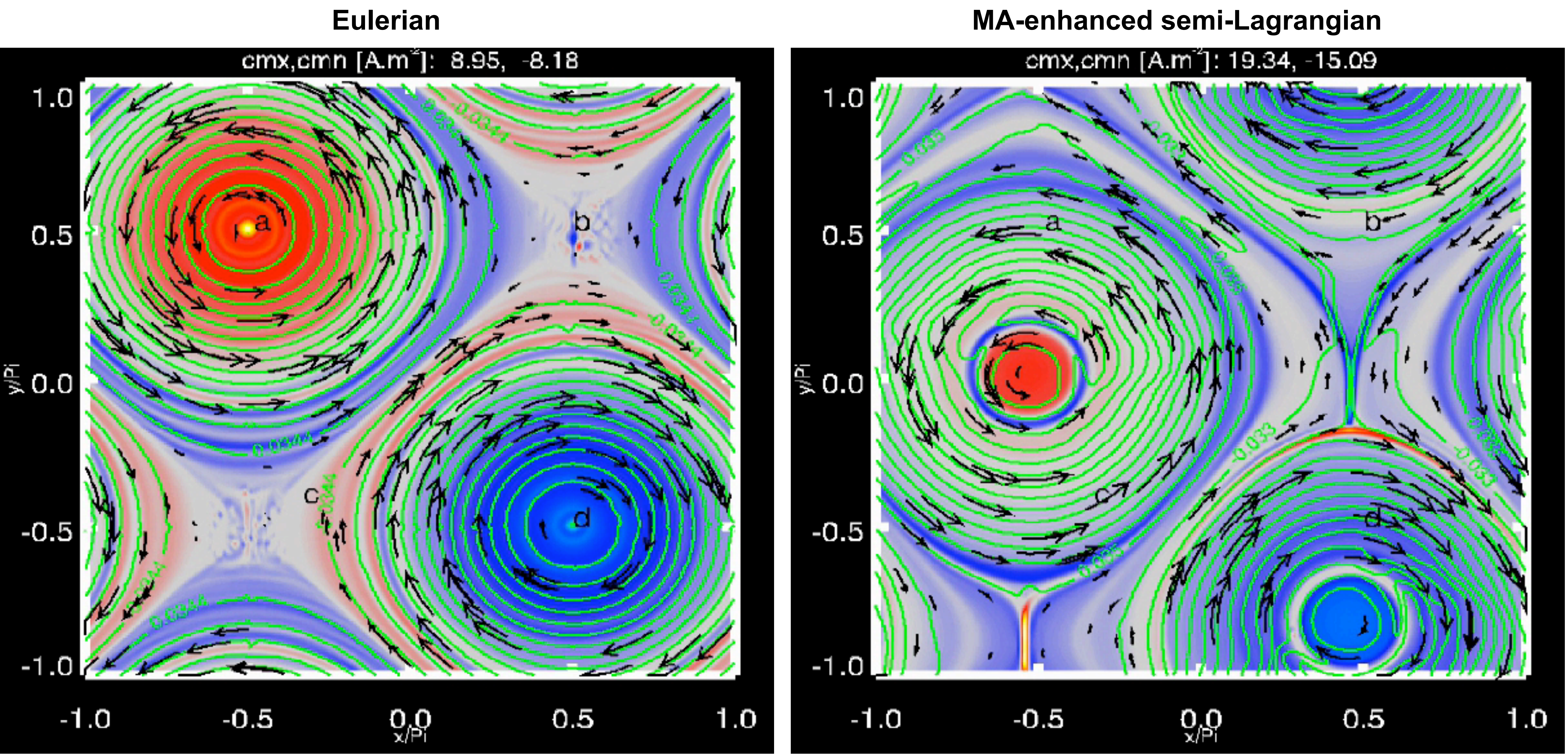
Classical SL solution generates spurious magnetic energy and violates Lagrangian mass conservation law!

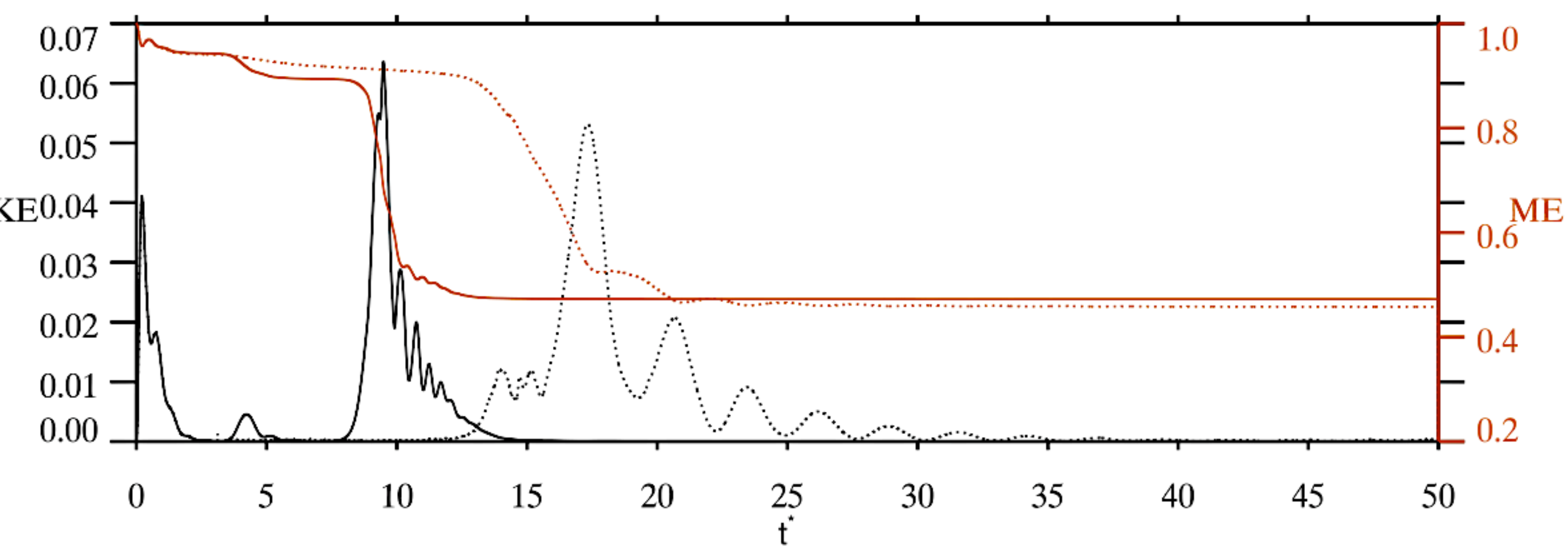


Solution (Cossette JF, Smolarkiewicz PK, Charbonneau P, JCP, in prep) : Apply the Monge-Ampère (MA) trajectory correction every timestep

$$(\tilde{\mathbf{x}}_0)_C = \tilde{\mathbf{x}}_0 + (t-t_0)\nabla\phi. \qquad \det\left\{\frac{\partial(\tilde{\mathbf{x}}_0)_C}{\partial\mathbf{x}}\right\} = 1$$

Nonlinear 2nd order PDE: Monge-Ampère equation


Eulerian
MA-enhanced semi-Lagrangian
cmx,cmn [A.m-2]: 8.95, -8.18
cmx,cmn [A.m-2]: 19.34, -15.09
a
b
c
d
y/Pi
x/Pi
1.0
0.5
0.0
-0.5
-1.0


KE
ME
t*
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
1.0
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30
35
40
45
50

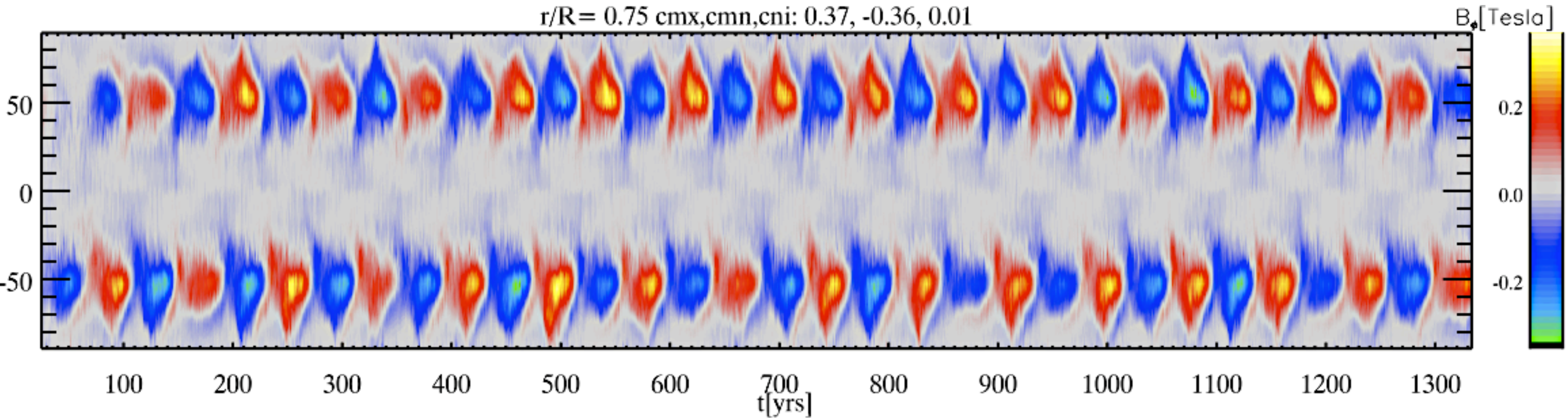
Conclusions and remarks

-Eulerian and SL solutions differ in many aspects: reconnection rate, morphology of the final equilibrium states, kinetic energy release, conservation of MHD invariants.

-Eulerian and SL advection operators select different dissipative paths to equilibrium. In other words, there is more than one possible final equilibrium state for the same initial condition.

-Grid resolution is key player in High R_m solutions obtained from DNS/LES. However, dissipative properties of the numerical scheme itself should not be overlooked.

So far, global MHD simulations of the solar SCZ have exclusively used Eulerian schemes.



How does this solution changes when using a MA-enhanced SL advection operator?