Session 4: **Helicity**
(Tues, May 22, 9-10:30 am and 11am-12:30 pm)

Rotation and Helicity in MHD Turbulence

John V. Shebalin, NASA JSC

Basic Equations

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{u} \times (\boldsymbol{\omega} + 2\boldsymbol{\Omega}_o) + \mathbf{j} \times (\mathbf{b} + \mathbf{B}_o)] + \nu \nabla^2 \boldsymbol{\omega},$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [\mathbf{u} \times (\mathbf{b} + \mathbf{B}_o)] + \eta \nabla^2 \mathbf{b}.$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \quad \mathbf{j} = \nabla \times \mathbf{b}.$$

$$\boldsymbol{\Omega}_o = \Omega_o \hat{\mathbf{z}} \text{ and } \mathbf{B}_o = B_o \hat{\mathbf{x}}$$

Fourier Representation

$$\begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \mathbf{b}(\mathbf{x},t) \end{bmatrix} = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \begin{bmatrix} \tilde{\mathbf{u}}(\mathbf{k},t) \\ \tilde{\mathbf{b}}(\mathbf{k},t) \end{bmatrix} e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\begin{bmatrix} \tilde{\mathbf{u}}(\mathbf{k},t) \\ \tilde{\mathbf{b}}(\mathbf{k},t) \end{bmatrix} = \frac{1}{N^{3/2}} \sum_{\mathbf{x}} \begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \mathbf{b}(\mathbf{x},t) \end{bmatrix} e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

$$\tilde{\mathbf{u}}(\mathbf{k}) = \tilde{u}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{u}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}),$$

$$\tilde{\mathbf{b}}(\mathbf{k}) = \tilde{b}_1(\mathbf{k},t)\hat{\mathbf{e}}_1(\mathbf{k}) + \tilde{b}_2(\mathbf{k},t)\hat{\mathbf{e}}_2(\mathbf{k}).$$

Modal expectation values

$$\langle \tilde{\mathbf{u}}(\mathbf{k}) \rangle = \langle \tilde{\mathbf{b}}(\mathbf{k}) \rangle = 0, \qquad \delta^2 = \alpha^2 - \tfrac{1}{4}\beta^2$$

$$E_k(\mathbf{k}) == \frac{2\alpha\left(\delta^2 - \gamma^2/k^2\right)}{\left(\delta^4 - \alpha^2\gamma^2/k^2\right)}, \qquad E_M(\mathbf{k}) = \frac{2\alpha\delta^2}{\left(\delta^4 - \alpha^2\gamma^2/k^2\right)}$$

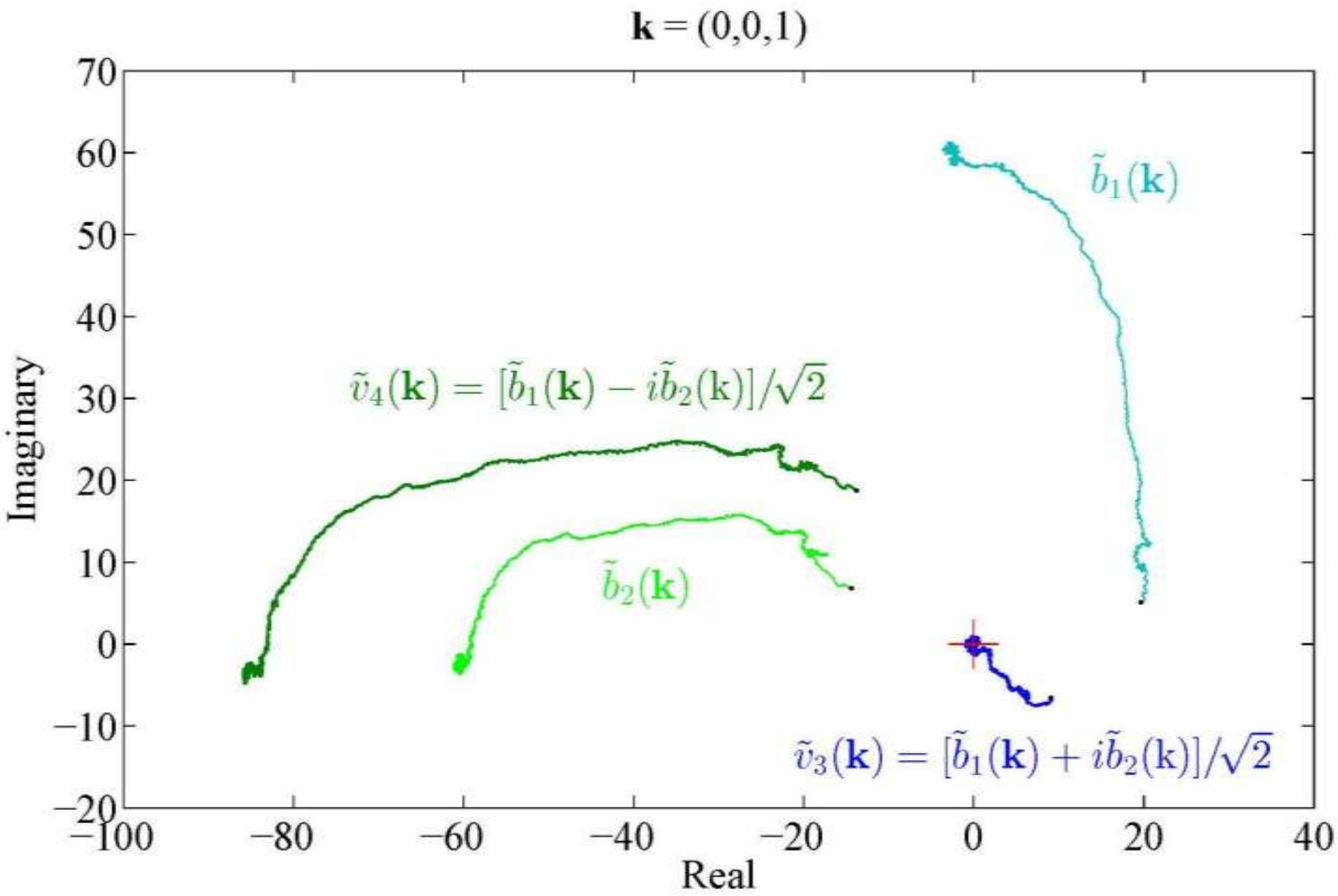
$$H_C(\mathbf{k}) = -\frac{\beta\delta^2}{\left(\delta^4 - \alpha^2\gamma^2/k^2\right)}, \qquad H_M(\mathbf{k}) = -\frac{2\alpha^2\gamma/k^2}{\left(\delta^4 - \alpha^2\gamma^2/k^2\right)}$$

Inverse Temperatures (E, H_M, H_C: ideal invariants)

$$\alpha = \frac{2r\langle E_M \rangle}{\langle E_M \rangle\left(\langle E \rangle - \langle E_M \rangle\right) - \langle H_C \rangle^2}, \qquad \frac{\beta}{\alpha} = -2\frac{\langle H_C \rangle}{\langle E_M \rangle}, \qquad \frac{\gamma}{\alpha} = \frac{\langle E \rangle - 2\langle E_M \rangle}{\langle H_M \rangle}$$

Coherent Structures in MHD Turbulence

32^3 Run R5-4; $\boldsymbol{\Omega}_o = 10\hat{\mathbf{z}}$, $\mathbf{B}_o = \mathbf{0}$



Broken Ergodicity: "In a system that is non-ergodic on physical timescales the phase point is effectively confined in one subregion or component of phase space."

Palmer (1982)

Volume Averages

$$\{\phi\psi\} \equiv (2\pi)^{-3} \int \phi(\mathbf{r},t)\psi(\mathbf{r},t) d^3x$$

$$= \frac{1}{N^3} \sum_{\mathbf{k}} \tilde{\phi}^*(\mathbf{k},t)\tilde{\psi}(\mathbf{k},t)$$

$$E_K = \tfrac{1}{2}\left\{u^2\right\}, \quad E_M = \tfrac{1}{2}\left\{b^2\right\}, \quad \Omega = \tfrac{1}{2}\left\{\omega^2\right\}, \quad J = \tfrac{1}{2}\left\{j^2\right\},$$

$$E = E_K + E_M, \quad H_C = \tfrac{1}{2}\{\mathbf{u}\cdot\mathbf{b}\}, \quad H_M = \tfrac{1}{2}\{\mathbf{a}\cdot\mathbf{b}\}.$$

Energy, Cross Helicity, Magnetic Helicity

Manipulate the basic MHD equations to get:

$$\frac{dE}{dt} = -2\left(\nu\Omega + \eta J\right),$$

$$\frac{dH_C}{dt} = \mathbf{\Omega}_{\mathrm{o}} \cdot \{\mathbf{b} \times \mathbf{u}\} - \tfrac{1}{2}\left(\nu + \eta\right)\{\mathbf{j} \cdot \boldsymbol{\omega}\},$$

$$\frac{dH_M}{dt} = \mathbf{B}_{\mathrm{o}} \cdot \{\mathbf{b} \times \mathbf{u}\} - \eta\,\{\mathbf{j} \cdot \mathbf{b}\}.$$

If $\nu = \eta = 0$ and $\mathbf{\Omega}_{\mathrm{o}} = \mathbf{B}_{\mathrm{o}} = 0$, E, H_M, H_C are ideal invariants.

Magnetic Vector Potential

$$\nabla \times \mathbf{a} = \mathbf{b}, \ \nabla \cdot \mathbf{a} = 0,$$

'Uncurl' the magnetic induction equation:

$$\frac{\partial}{\partial t}(\mathbf{a} + \mathbf{A}) = -\nabla\Phi + \mathbf{u} \times (\mathbf{b} + \mathbf{B}_\circ) + \eta\nabla^2\mathbf{a}.$$

Here $\mathbf{a}(\mathbf{x}, t)$ satisfies $\{\mathbf{a}\} = \mathbf{0}$, while $\mathbf{A}(t) = \{\mathbf{A}\}$

$$\frac{d\mathbf{A}}{dt} = \{\mathbf{u} \times \mathbf{b}\} \rightarrow \mathbf{A}(t) = \mathbf{A}_\circ + \int_0^t \{\mathbf{u} \times \mathbf{b}\} dt'.$$

Generalized Helicities

$$\frac{dE}{dt} = \frac{dG_C}{dt} = \frac{dG_M}{dt} = 0.$$

The ideal invariants G_C and G_M are, respectively, generalized cross and magnetic helicities:

$$G_C \equiv H_C(t) + \mathbf{\Omega}_o \cdot \mathbf{A}(t),$$

$$G_M \equiv H_M(t) + \mathbf{B}_o \cdot \mathbf{A}(t).$$

E, G_C and G_M are ideal invariants even in the case where either $\mathbf{\Omega}_o \neq \mathbf{0}$ or $\mathbf{B}_o \neq \mathbf{0}$.

Ideal Invariants, Generalized

$$G_C = H_C(t) + \mathbf{\Omega}_\mathrm{o} \cdot \mathbf{A}(t) = H_C(0) + \mathbf{\Omega}_\mathrm{o} \cdot \mathbf{A}_\mathrm{o},$$

$$G_M = H_M(t) + \mathbf{B}_\mathrm{o} \cdot \mathbf{A}(t) = H_M(0) + \mathbf{B}_\mathrm{o} \cdot \mathbf{A}_\mathrm{o}.$$

$\mathbf{A}_\mathrm{o}$ can always be chosen so that

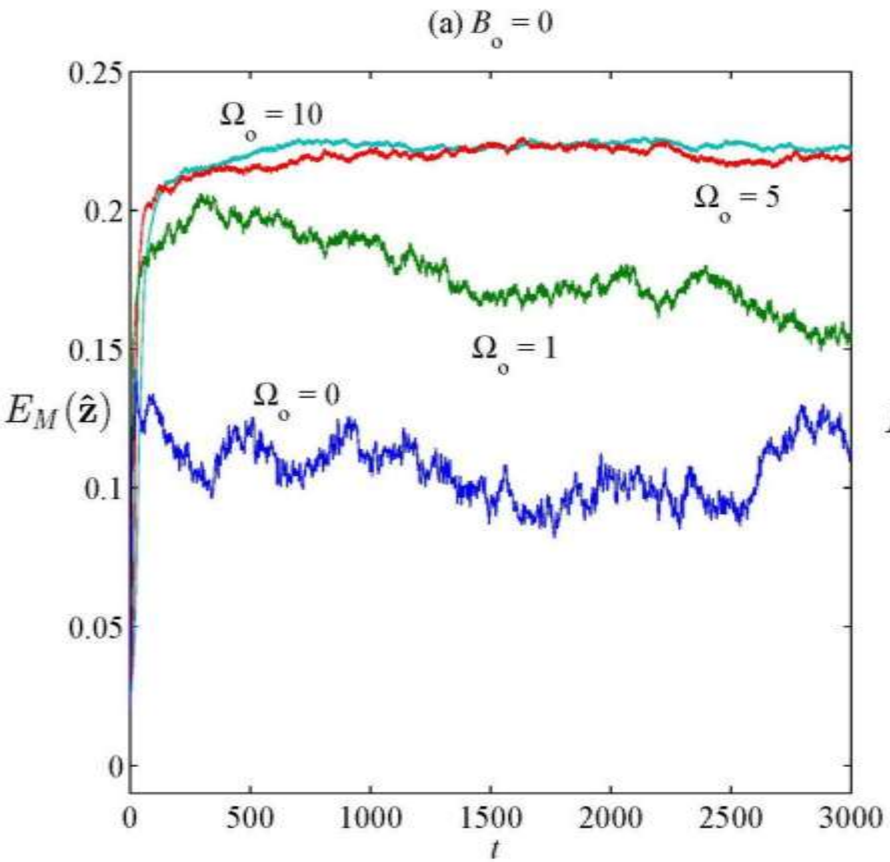
$$G_C = 0 \text{ when } \mathbf{\Omega}_\mathrm{o} \neq \mathbf{0} \text{ and } G_M = 0 \text{ when } \mathbf{B}_\mathrm{o} \neq \mathbf{0}.$$

Numerically, the rms values of $H_C(t)$ and $H_M(t)$ appear independent of the values of Ω_o and B_o; thus,

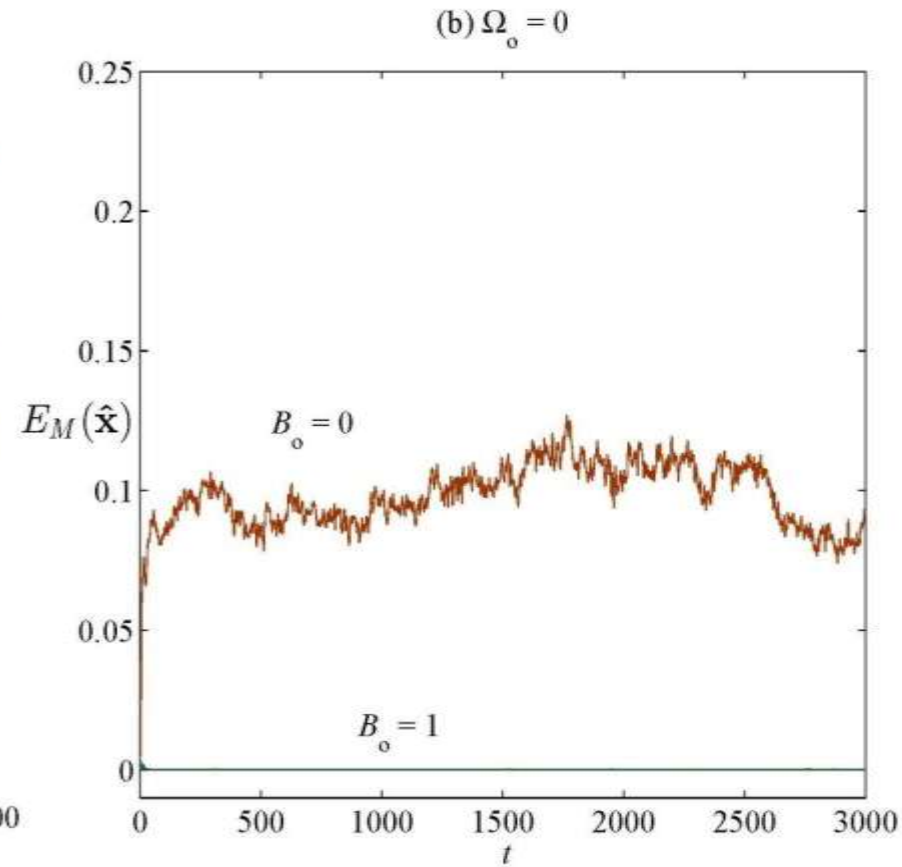
$$\lim_{\Omega_\mathrm{o}\to\infty} \left| \hat{\mathbf{z}} \cdot \int_{t_1}^{t_2} \{\mathbf{u} \times \mathbf{b}\}\, dt \right| = 0, \quad \lim_{B_\mathrm{o}\to\infty} \left| \hat{\mathbf{x}} \cdot \int_{t_3}^{t_4} \{\mathbf{u} \times \mathbf{b}\}\, dt \right| = 0.$$

Depression of Nonlinearity

$$\lim_{\Omega_o \to \infty} \left| \hat{\mathbf{z}} \cdot \int_{t_1}^{t_2} \{\mathbf{u} \times \mathbf{b}\} \, dt \right| = 0, \quad \lim_{B_o \to \infty} \left| \hat{\mathbf{x}} \cdot \int_{t_3}^{t_4} \{\mathbf{u} \times \mathbf{b}\} \, dt \right| = 0.$$



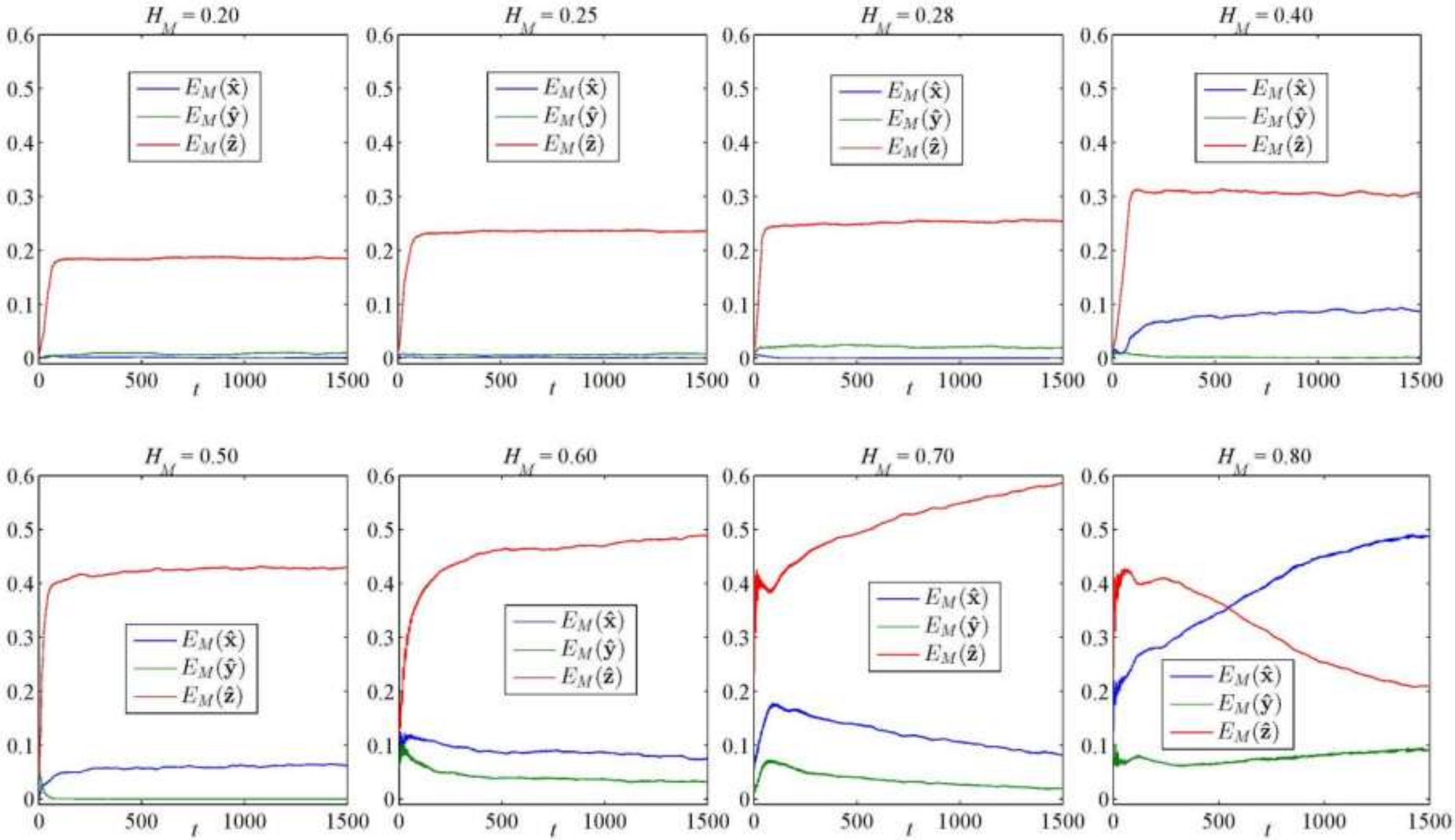
Dipole alignment, anisotropy, …



Dynamic alignment, …

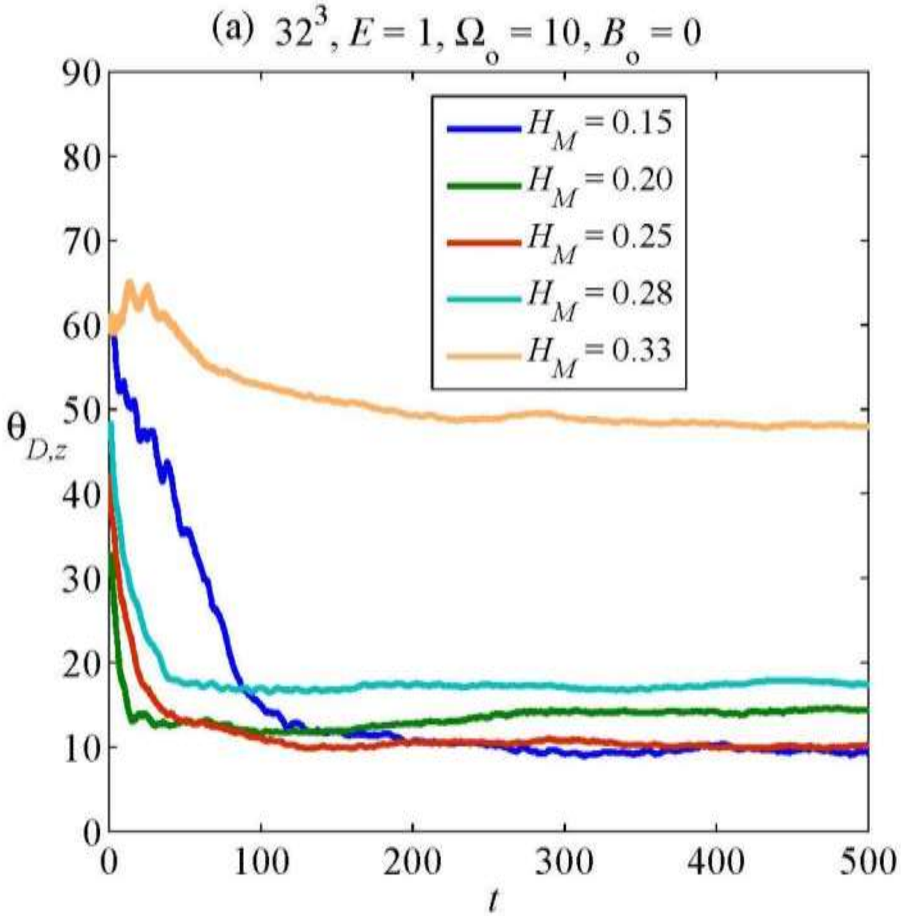
Magnetic Energy at $k = 1$ vs H_M

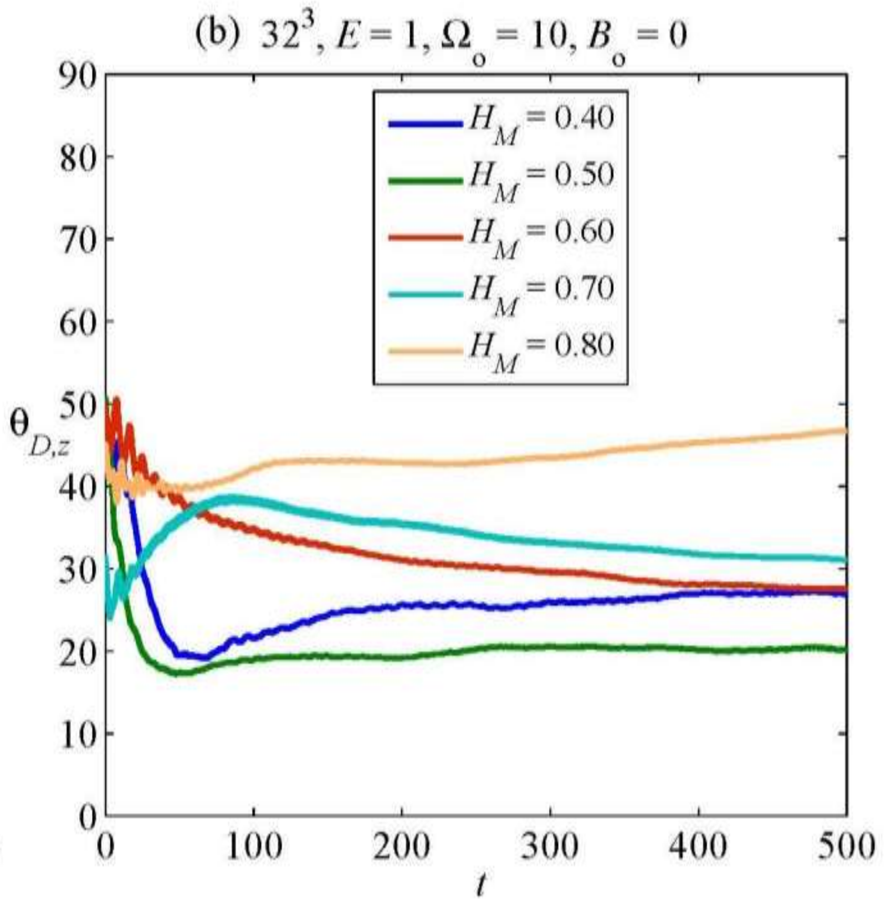
Runs, Set 4; $\mathbf{\Omega}_o = 10\hat{\mathbf{z}}$



Dipole Angle vs H_M

Runs, Set 4; $\mathbf{\Omega}_o = 10\hat{\mathbf{z}}$





Discussion

- Ideal MHD: for $\mathbf{B}_o = 0$, $H_C(\mathbf{k}) >> H_M(\mathbf{k})$ as $k \to \infty$.
- Implications for SGS?
- Coherent structure due to broken ergodicity occurs at low k.
- Implications for the LE part of LES?
- Coherent energy increases with H_M; less dependent on H_C.
- Definition of cross and magnetic helicities can be generalized.
- Imply decrease in fluctuations with k in $\mathbf{\Omega}_o$ or $\mathbf{B}_o$ directions.