Session 5: **Geometry and Boundary Conditions**
(Tues, May 22, 2-3:30 pm and 4am-5:30 pm)

Ideal MHD Turbulence in Cartesian and Spherical Geometries

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Boundary Conditions

- No-slip; free-slip; **B** continuous; B_r continuous; no-stress, *etc*.
- External medium: vacuum, insulator, (super)conductor, … .
- All b.c.s are approximate and usually chosen for convenience.
- For Cartesian geometry b.c.s, **u**, **b**, *etc*., are periodic.
- For the spherical geometry, an expedient set of b.c.s is:

$$\text{Homogeneous b.c.s}: \begin{cases} \hat{\mathbf{r}}\cdot\mathbf{u}\big|_{r_1,r_2} = \hat{\mathbf{r}}\cdot\mathbf{j}\big|_{r_1,r_2} = 0, \\ \hat{\mathbf{r}}\cdot\boldsymbol{\omega}\big|_{r_1,r_2} = \hat{\mathbf{r}}\cdot\mathbf{b}\big|_{r_1,r_2} = 0. \end{cases}$$

(Mininni & Montgomery, 2006, 2007)

$$\text{External field}: \quad \hat{\mathbf{r}}\times(\mathbf{b}-\mathbf{B}_{ext})\big|_{r_1,r_2} = 0$$

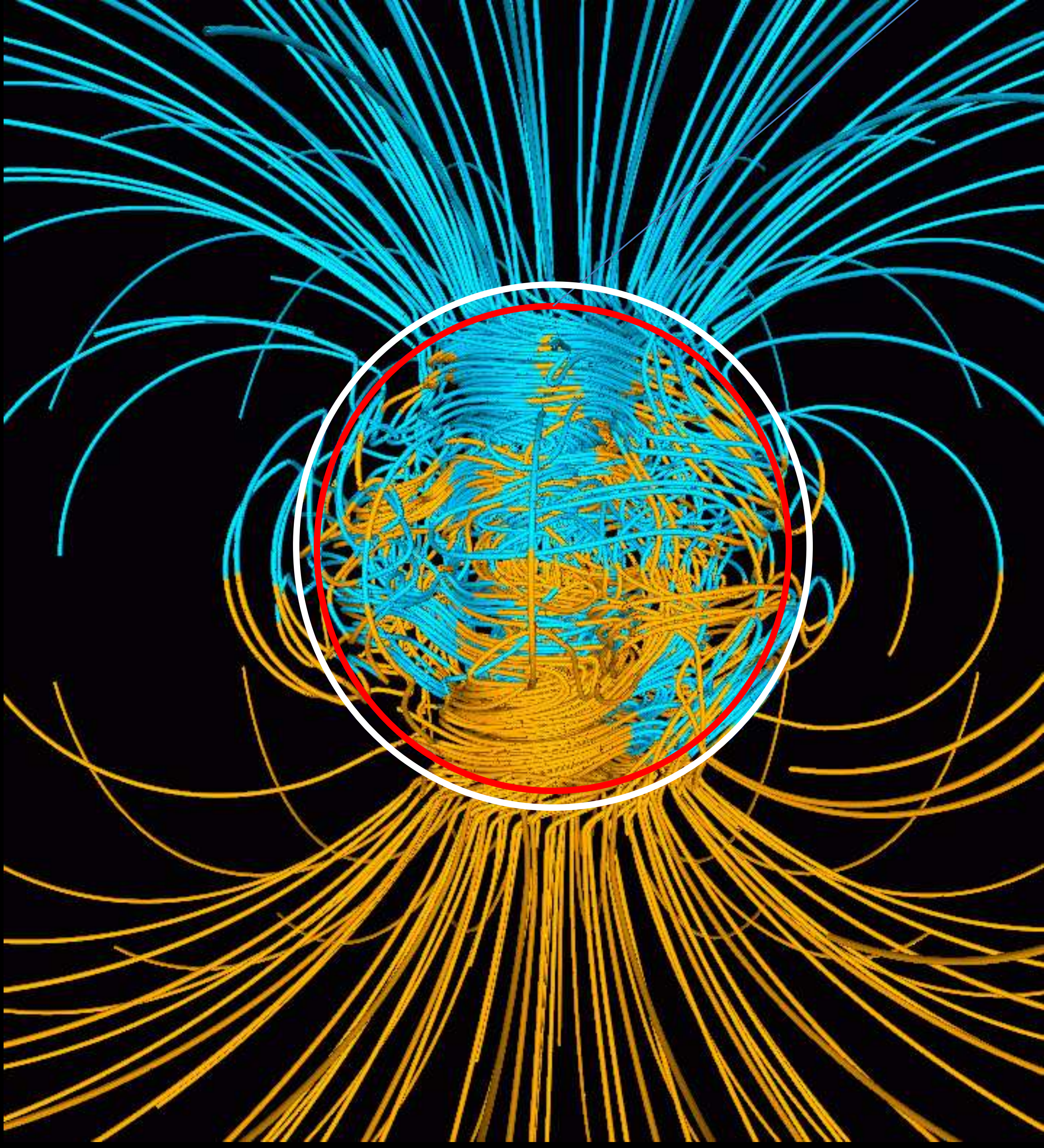
(Shebalin, 2012)

Geodynamo Simulations

The Glatzmaier-Roberts geodynamo numerical model.*,† Magnetic field lines are blue where the field is directed inward and yellow where directed outward. The rotation axis of the model Earth is vertical and through the center.

– core-mantle boundary

– homogeneity surface



*Glatzmaier & Roberts, *Nature* **377**, 203-209 (1995).

†Glatzmaier & Roberts, *Phys. Earth Planet. Int.* **91**, 63-75 (1995).

Galerkin Expansions

(Mininni & Montgomery, 2006, 2007)

Spherical case: $\boldsymbol{S}(b_{lmn}, a_{lmn}) \equiv \sum_{l,m,n} [b_{lmn}(t)\boldsymbol{T}_{lmn}(\boldsymbol{x}) + a_{lmn}(t)\boldsymbol{P}_{lmn}(\boldsymbol{x})]$

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{S}(u_{lmn}, w_{lmn}), \quad \boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{S}(b_{lmn}, a_{lmn})$$

$$\boldsymbol{T}_{lmn}(\boldsymbol{x}) = \hat{g}_l(k_{ln} r)\hat{\boldsymbol{\Phi}}_{lm}(\theta, \varphi), \; \nabla \times \boldsymbol{T}_{lmn}(\boldsymbol{x}) = \boldsymbol{P}_{lmn}(\boldsymbol{x})$$

$\boldsymbol{T}_{lmn}(\boldsymbol{x})$ and $\boldsymbol{P}_{lmn}(\boldsymbol{x})$ satisfy homogeneous b.c.s at $r_1 = 1$ and $r_2 = R_2/R_1$. The $\hat{g}_l(z)$ are linear combinations of the spherical Bessel & Neumann functions $j_l(z)$ and $n_l(z)$ such that $\hat{g}_l(z_1) = \hat{g}_l(z_2) = 0$, $z_1 = k_{ln}$, $z_2 = k_{ln} r_2$.

Cartesian case:

(periodic b.c.s)

$$\begin{bmatrix} \mathbf{u}(\mathbf{x}, t) \\ \mathbf{b}(\mathbf{x}, t) \end{bmatrix} = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \begin{bmatrix} \tilde{\mathbf{u}}(\mathbf{k}, t) \\ \tilde{\mathbf{b}}(\mathbf{k}, t) \end{bmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}$$

Mind-Numbing Details

$$\boldsymbol{T}_{lmn}(\boldsymbol{x}) = \hat{g}_l(k_{ln}r)\hat{\boldsymbol{\Phi}}_{lm}(\theta,\varphi), \quad 1 \le l \le L, \quad -l \le m \le l, \quad 1 \le n \le N,$$

$$\boldsymbol{P}_{lmn}(\boldsymbol{x}) = -\frac{1}{r}\left[\sqrt{l(l+1)}\,\hat{g}_l(k_{ln}r)\hat{\boldsymbol{Y}}_{lm}(\theta,\varphi) + \frac{\mathrm{d}}{\mathrm{d}r}[r\hat{g}_l(k_{ln}r)]\hat{\boldsymbol{\Psi}}_{lm}(\theta,\varphi)\right].$$

$$\hat{\boldsymbol{Y}}_{lm}(\theta,\varphi) = \hat{\boldsymbol{r}}\mathrm{Y}_{lm}(\theta,\varphi), \;\; \hat{\boldsymbol{\Phi}}_{lm}(\theta,\varphi) = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{\Psi}}_{lm}(\theta,\varphi)$$

$$\hat{\boldsymbol{\Psi}}_{lm}(\theta,\varphi) = \frac{r}{\sqrt{l(l+1)}}\boldsymbol{\nabla}\mathrm{Y}_{lm}(\theta,\varphi),$$

$$\hat{g}_l(k_{ln}r) = N_{ln}^{-1}g_l(k_{ln}r), \quad 1 \le r \le r_\circ = R_2/R_1,$$

$$g_l(k_{ln}r) = n_l(k_{ln}r_\circ)j_l(k_{ln}r) - j_l(k_{ln}r_\circ)n_l(k_{ln}r),$$

$$N_{ln}^2 = \frac{1}{2k_{ln}^4}\left(\frac{1}{r_\circ} - \left[\frac{n_l(k_{ln}r_\circ)}{n_l(k_{ln})}\right]^2\right),$$

$$g_l(k_{ln}) = 0, \quad n = 1, 2, \ldots, N; \;\; l = 1, 2, \ldots, L.$$

Ideal MHD Stat Mech

The PDF* D is based on the ideal invariants $E = ½\int(u^2+b^2)dV$, $H_C = ½\int\mathbf{u}\cdot\mathbf{b}dV$ and $H_M = ½\int\mathbf{a}\cdot\mathbf{b}dV$, and can be written as

$$D = \prod_{\mathcal{K}} D_{lmn}, \quad Z = \prod_{\mathcal{K}} Z_{lmn}, \quad \mathrm{d}\Gamma = \prod_{\mathcal{K}} \mathrm{d}\Gamma_{lmn},$$

$$D_{lmn} = \frac{\exp[-Q_{lmn}]}{Z_{lmn}}, \quad Z_{lmn} = \int \exp[-Q_{lmn}]\mathrm{d}\Gamma_{lmn}.$$

$$\begin{aligned} Q_{lmn} = {} & \alpha c_m\left(|u_{lmn}|^2 + k_{ln}^2|w_{lmn}|^2 + |b_{lmn}|^2 + k_{ln}^2|a_{lmn}|^2\right) \\ & + c_m \mathrm{Re}\left\{\beta\left(u_{lmn}^* b_{lmn} + k_{ln}^2 w_{lmn}^* a_{lmn}\right) + 2\gamma a_{lmn}^* b_{lmn}\right\}. \end{aligned}$$

*PDF = Probability Density Function

Ideal MHD Expectation Values

$$\langle |u_{lmn}|^2 \rangle = k_{ln}^2 \langle |w_{lmn}|^2 \rangle = \frac{\alpha(\delta^2 - \gamma^2/k_{ln}^2)}{\delta^4 - \alpha^2\gamma^2/k_{ln}^2},$$

$$\langle |b_{lmn}|^2 \rangle = k_{ln}^2 \langle |a_{lmn}|^2 \rangle = \frac{\alpha\delta^2}{\delta^4 - \alpha^2\gamma^2/k_{ln}^2},$$

$$\langle u_{lmn}^* b_{lmn} \rangle = k_{ln}^2 \langle w_{lmn}^* a_{lmn} \rangle = -\frac{\beta}{2\alpha} \langle |b_{lmn}|^2 \rangle,$$

$$\langle a_{lmn}^* b_{lmn} \rangle = \frac{\alpha}{\gamma} \langle |u_{lmn}|^2 - |b_{lmn}|^2 \rangle.$$

Spherical $\rightarrow$ Cartesian case (periodic box) by substituting

$k_{ln} \rightarrow k = |\mathbf{k}|$ and b_{lmn} , $a_{lmn} \rightarrow b_1(\mathbf{k}), b_2(\mathbf{k})$; *etc.*

Discussion

- Statistical analysis is based on Galerkin expansions.
- ‘Dynamic alignment’ $\rightarrow$ homogeneous spherical b.c.s.
- External magnetic field, $\mathbf{B}_{ext} = -\nabla\varphi_M$: $\hat{\mathbf{r}}\times(\mathbf{b} - \mathbf{B}_{ext}) = 0$.
- Homogeneous spherical ~ periodic box ideal MHD stat mech.
- Same features (coherent structure, *etc*.) are expected in both.
- Equivalence allows simpler, faster Cartesian DNS to serve as a surrogate for spherical DNS for testing SGS models, *etc*.
- A spherical geometry Galerkin DNS could be run in order to confirm equivalence with Cartesian periodic box case.