Asymptotic Approaches for Rotationally Constrained Convective Flows

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Navier-Stokes Equations: GAFD

- Generic non-dimensionalization: $L, U, \Delta T, P$

$$D_t \boldsymbol{u} + Ro^{-1} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla p + \Gamma T \widehat{\boldsymbol{r}} + Re^{-1} \nabla^2 \boldsymbol{u} + \boldsymbol{S}$$

$$D_t T = Pe^{-1} \nabla^2 T$$

$$\nabla \cdot \boldsymbol{u} = 0$$

where $D_t := \partial_t + \boldsymbol{u} \cdot \nabla$ with $(\boldsymbol{u}, p, T)$ for velocity, pressure & temperature fields.

- Non-dimensional Parameters:

Rossby Number $Ro = \dfrac{U}{2\Omega L}$

Euler Number $Eu = \dfrac{P}{\rho_0 U^2}$

Buoyancy Number $\Gamma = \dfrac{g\alpha \Delta T L}{U^2}$

Ekman Number $Ek = \dfrac{Ro}{Re} = \dfrac{\nu}{2\Omega L^2}$

Reynolds Number $Re = \dfrac{UL}{\nu}$

Péclet Number $Pe = \dfrac{UL}{\kappa}$

Navier-Stokes Equations: Rotationally Constrained Flows, $Ro \ll 1$

- For $Ro \ll 1$ turbulence challenge compounded

$$\boxed{D_t \boldsymbol{u} + Ro^{-1} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla p} + \Gamma T \widehat{\boldsymbol{r}} + Re^{-1} \nabla^2 \boldsymbol{u} + \boldsymbol{S}$$

$$D_t T = Pe^{-1} \nabla^2 T$$

$$\nabla \cdot \boldsymbol{u} = 0$$

- NSE stiff PDE, $\exists$ fast inertial waves & slow geostrophically balanced eddies

Fast Inertial Waves

$$\omega_{fast} \sim Ro^{-1} \frac{k_z}{\sqrt{k_\perp^2 + k_z^2}}$$

of secondary importance

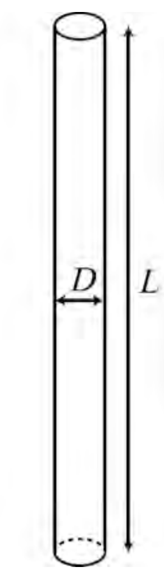
Geostrophic Eddies/Slow Waves

$$\omega_{slow} \sim \mathcal{O}(1)$$

$$Ro^{-1} \widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu \nabla p, \quad \nabla \cdot \boldsymbol{u} = 0 \Rightarrow$$

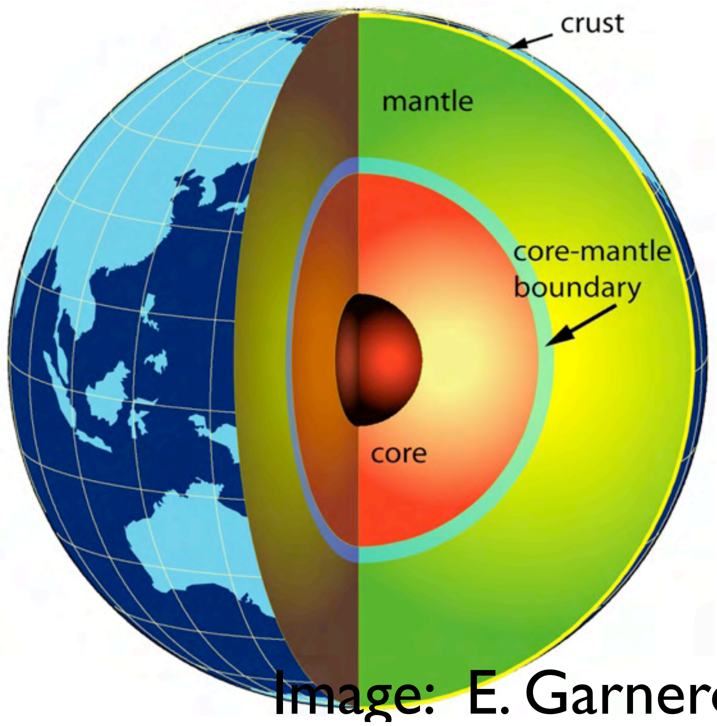
$$\widehat{\boldsymbol{z}} \cdot \nabla (\boldsymbol{u}, p) \approx 0$$

Proudman-Taylor Thm (1916,1923)
motions are inherently columnar



Planetary Scale Rotationally Constrained Convection $Ro \ll 1$

turbulence primary driver for geomagnetic field



$Ro \sim 10^{-7}$

$Re \sim 10^{8}$

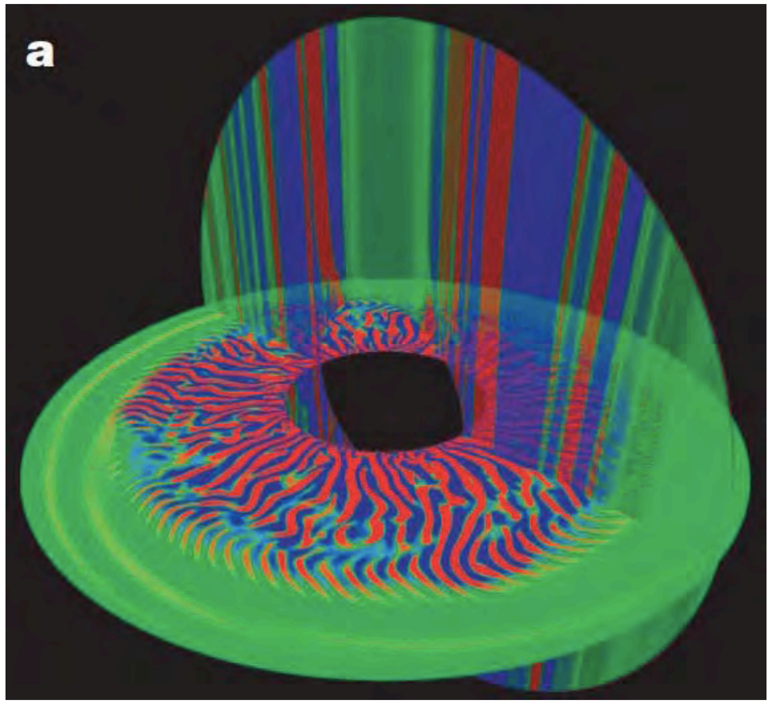
$Ek \sim 10^{-15}$

large-scale flow generation

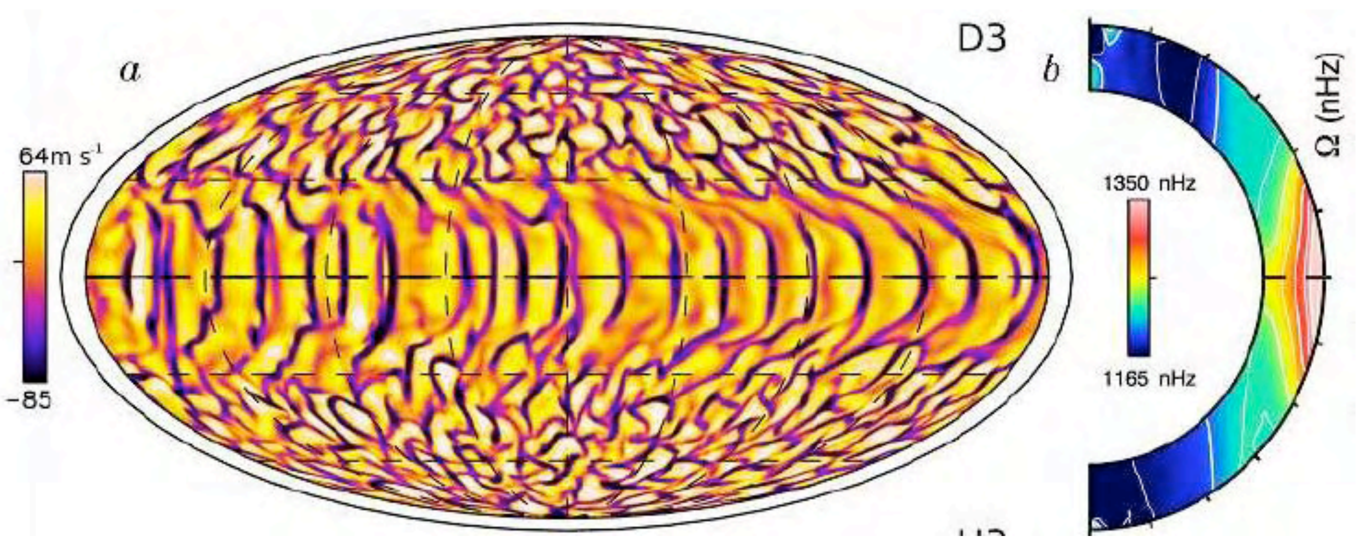
$Ro \sim 10^{-2}$

$Re \sim 10^{16}$

$Ek \sim 10^{-18}$



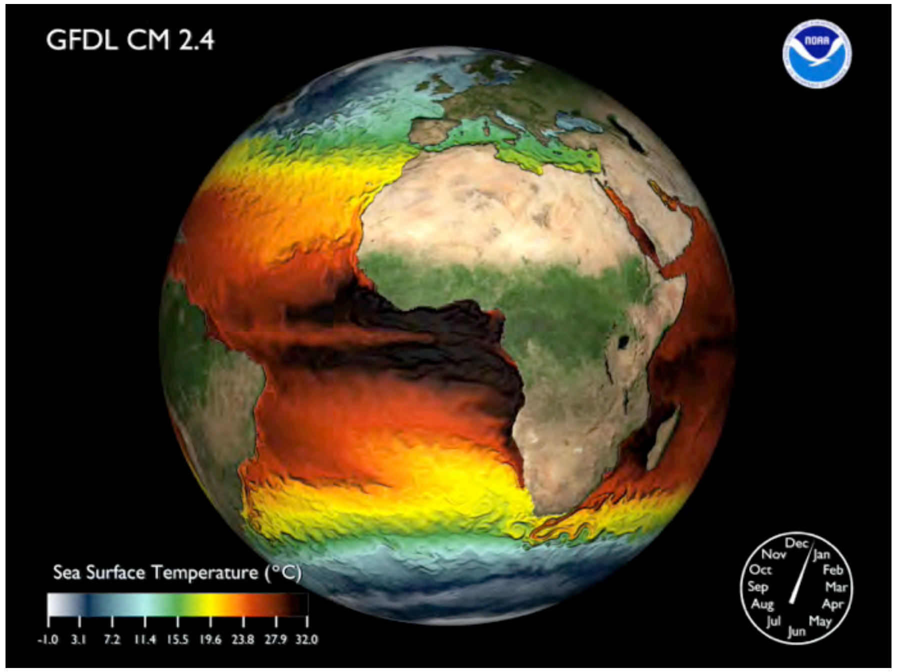
Axial vorticity, Ek~1e-6 (Kageyama et al Nature 2008)
Convective Rossby waves, still viscously controlled
Earth's Core: $l \sim DE^{1/3} \approx 10m$



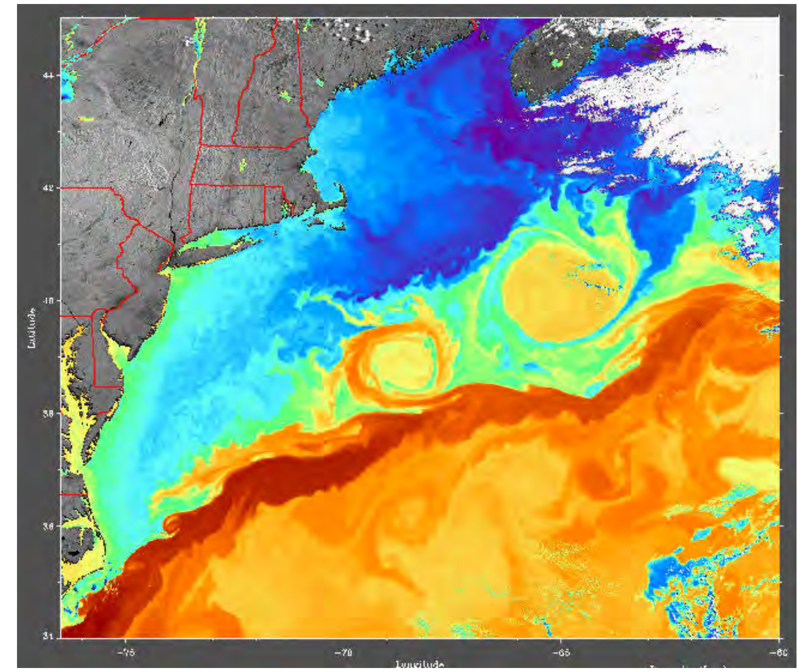
Rapidly rotating Sun; Brown et al ApJ 2010

Ocean dynamics

planetary - gyre scale ~ O(1000) km

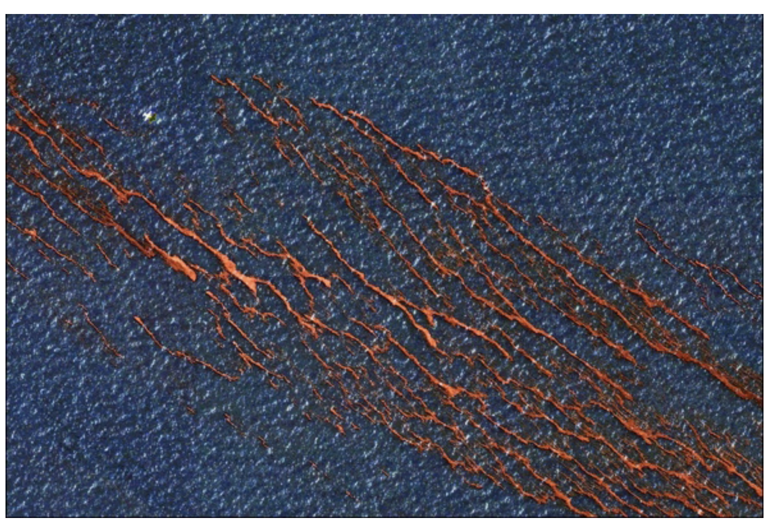


mesoscale ~ O(100) km

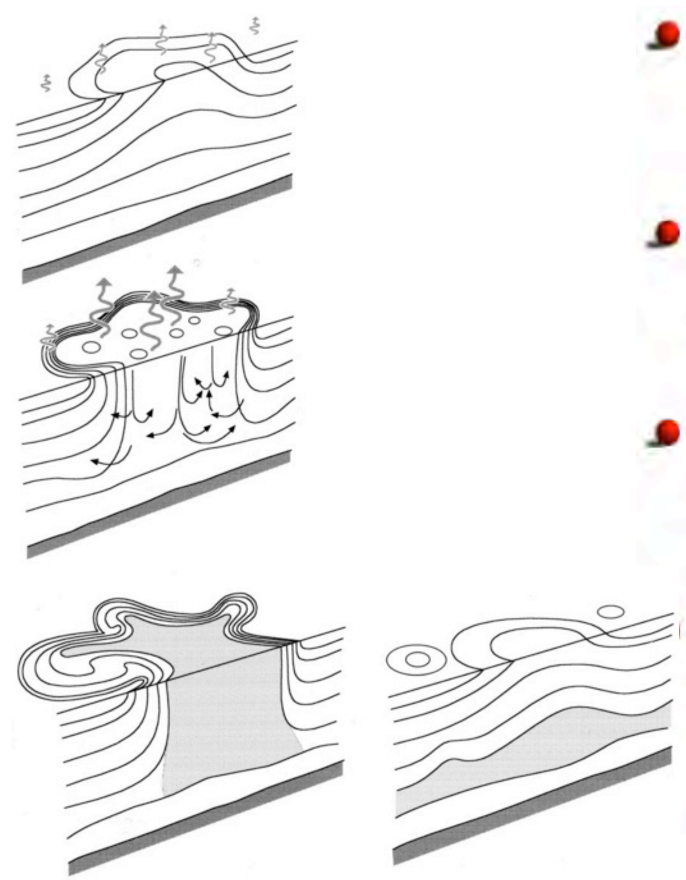


submesoscale ≤ O(1) km

Langmuir Turbulence

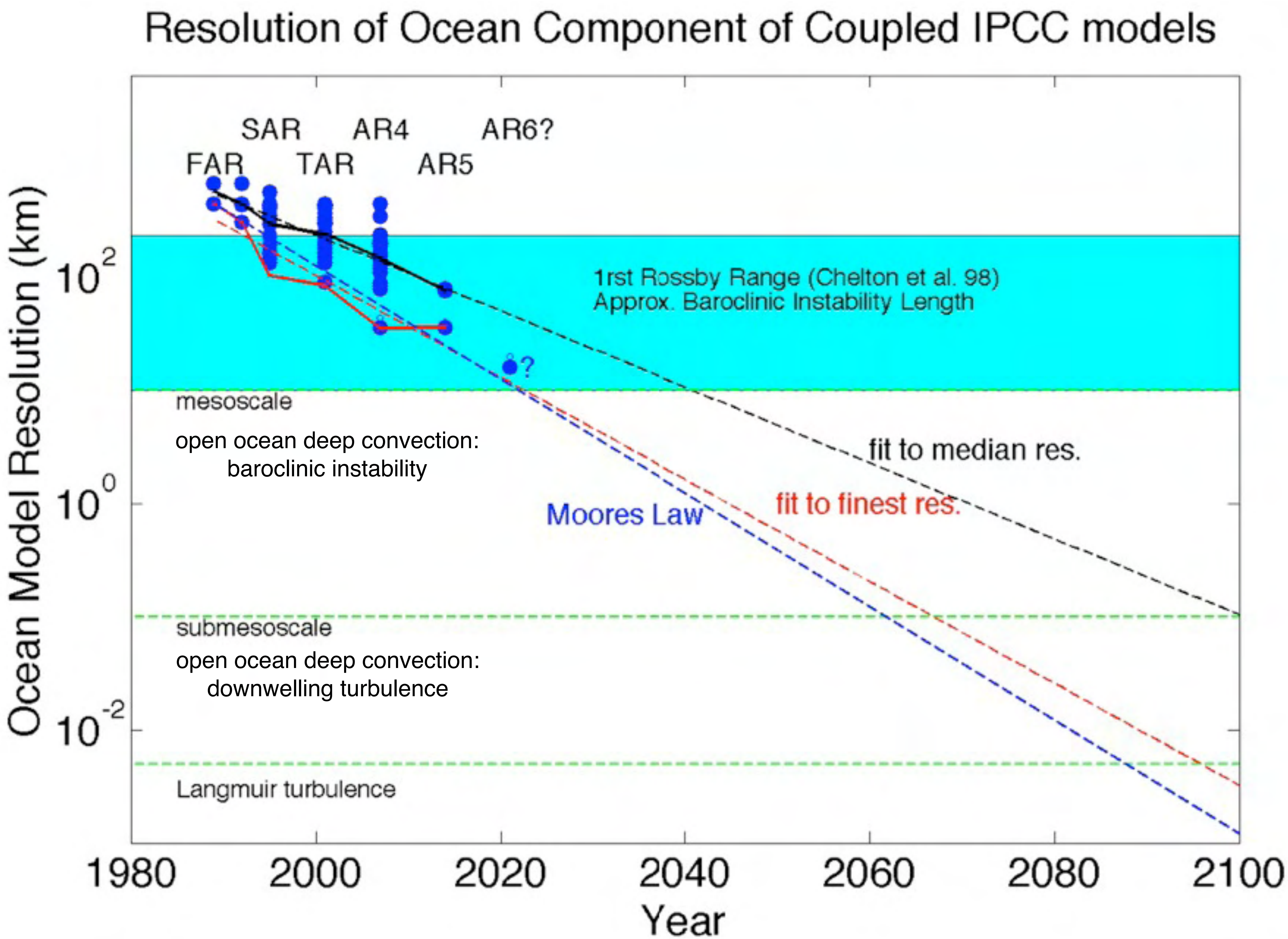


open-ocean deep convection: mesoscale

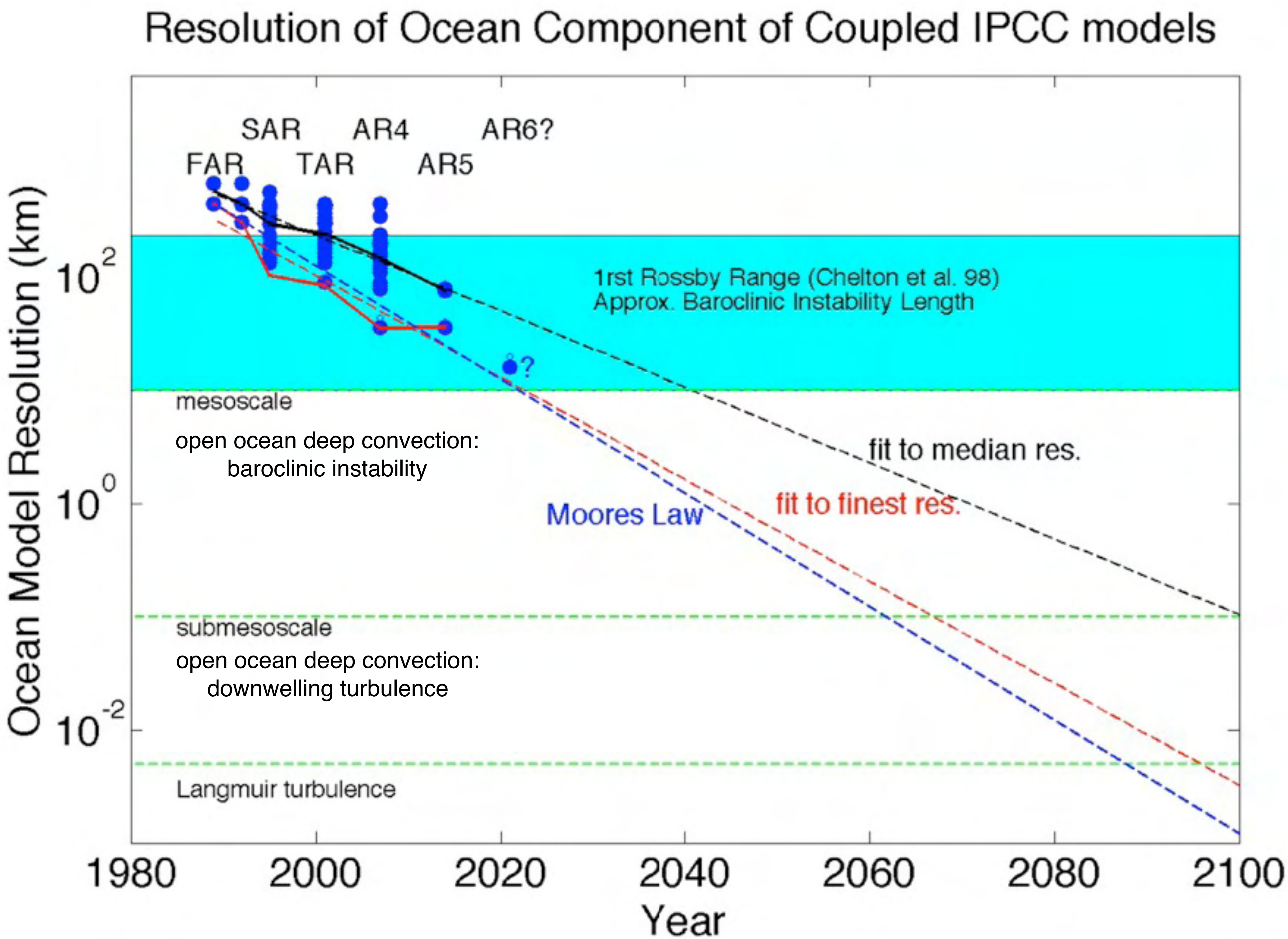


- preconditioning
  - cyclonic gyre domes isopycnals, $L \sim 100$km
- deep convection
  - cooling events trigger deep plumes, $L \lesssim 1$km $H \sim 2$km, $U \lesssim 10$ cm/s
- lateral exchange
  - geostrophic eddies, $L \sim 10$km
- influenced by rotation
  - natural Rossby number $Ro^* \sim 0.1 - 0.4$

$$Ro^* = \frac{L_{rot}}{H} = \left(\frac{B}{f^3 H^2}\right)^{1/2}$$



Top-down approach: DNS not possible for several centuries!



Approach to geostrophic turbulence bottom-up? .... middle-out?

Rotationally constrained (geostrophic) convective flows are highly anisotropic

- When do geometry and boundary conditions directly influence the small scales, and how might this be parameterized
  - theory ⇒ mechanical bc's of secondary importance; Lab/DNS ⇒ hard to achieve high Re - Low Ro regimes
- When do boundary conditions influence the small scales via the dynamics of the large scales
  - vortex stretching in spherical geometries?
- Differences for LES closures to address in cartesian and spherical geometries
  - presently N/A
- When are boundary conditions unimportant?
  - thermal bl's have surprising affect on rotational constraint
- Links between small scale and large scales
  - geostrophic convective turbulence very efficient at driving large-scale flows, sustained in a +ve feedback loop
- Where is the KE injected? Should the buoyancy force do significant work on the SGS?
  - to-date KE spectrum containing convection must be resolved, Low Ro challenge ⇒ separation of scales issue
- Magnetic Field/dynamos!

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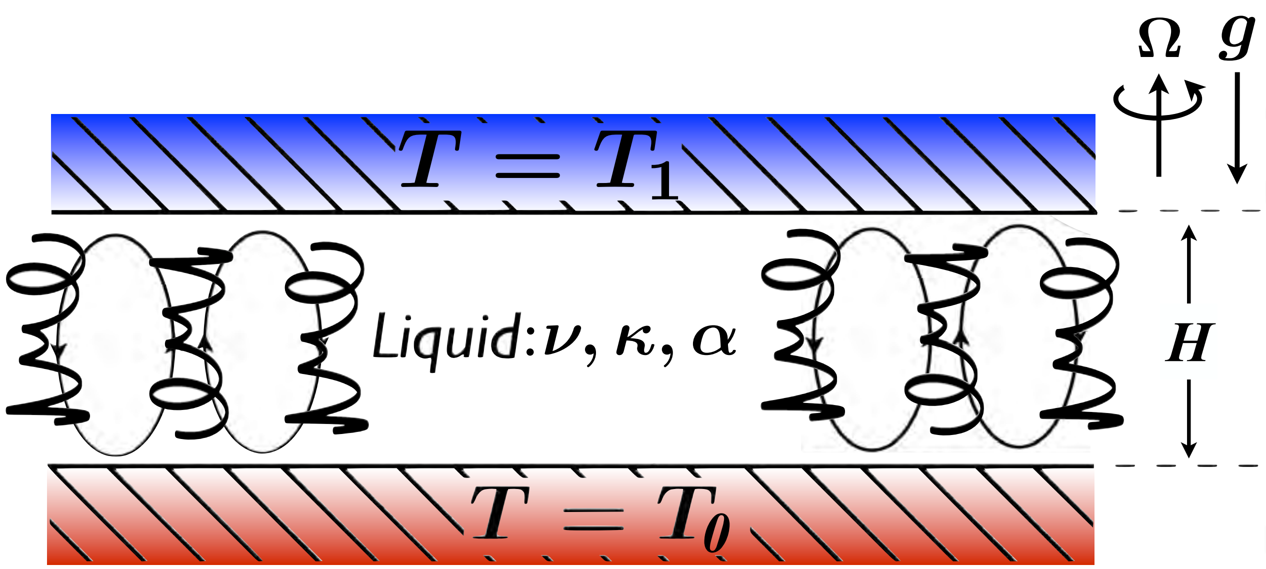
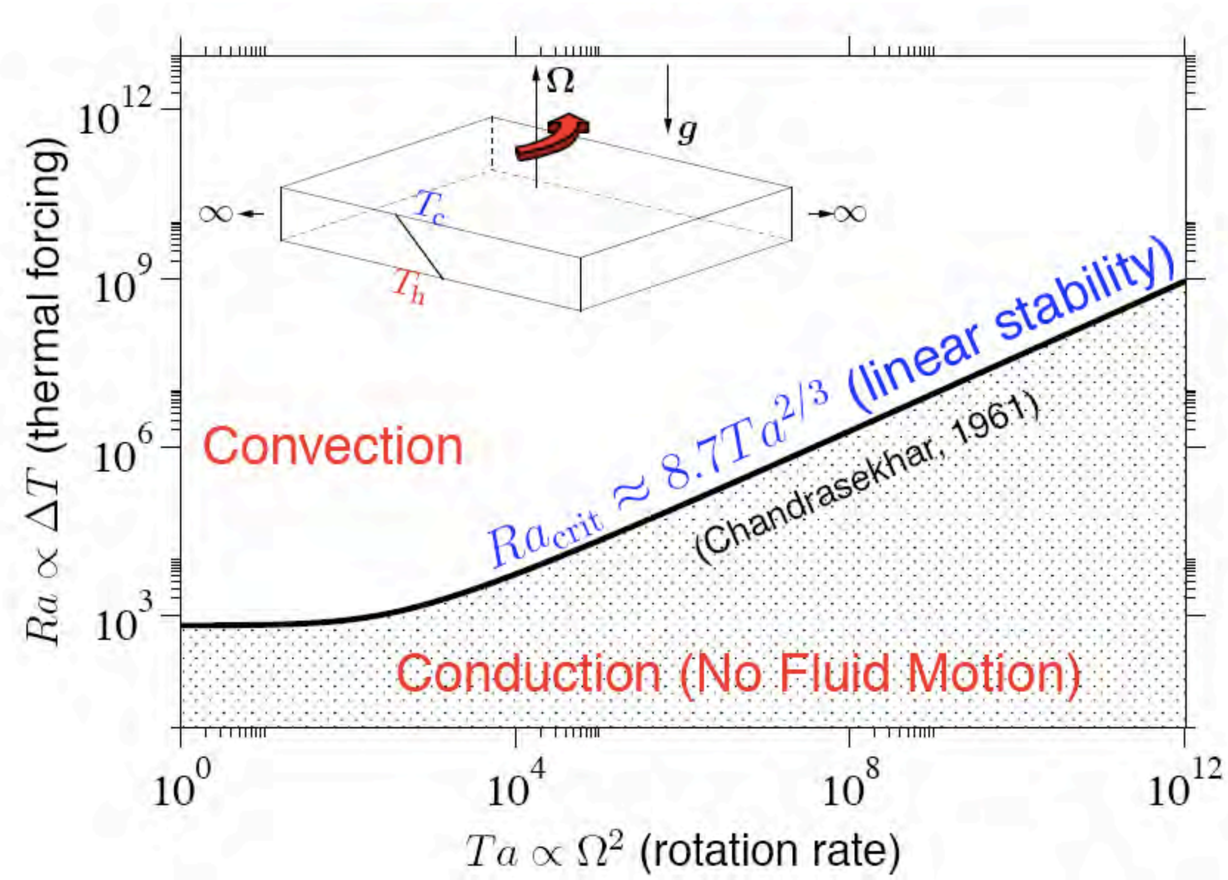
Investigate simplified model scenarios

Rotating Rayleigh-Bénard Convection





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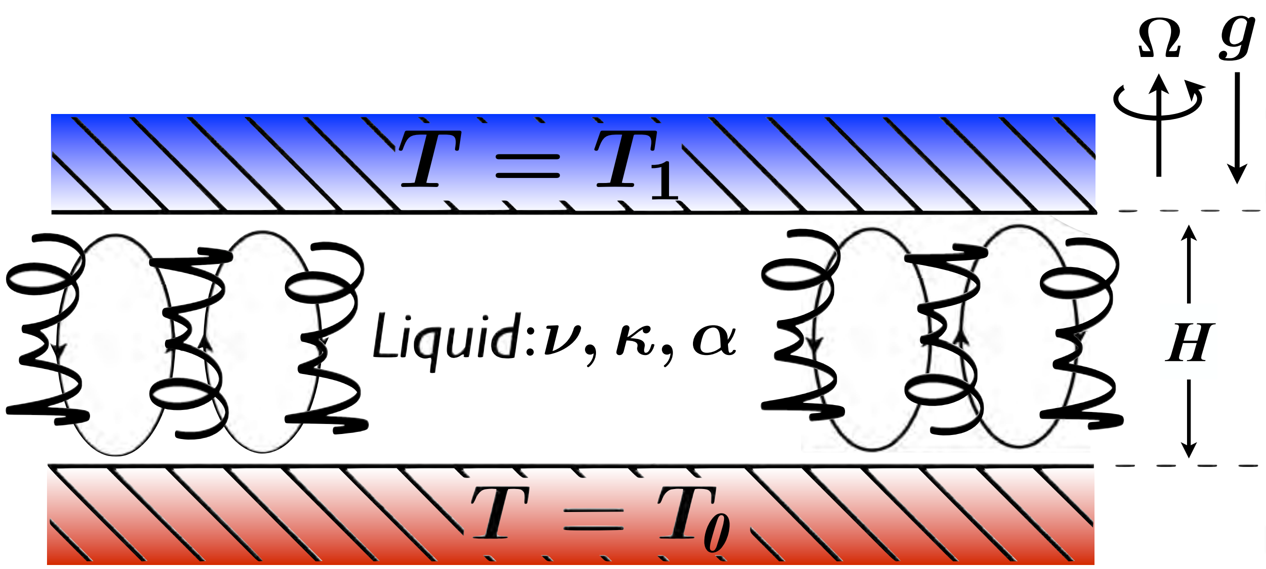

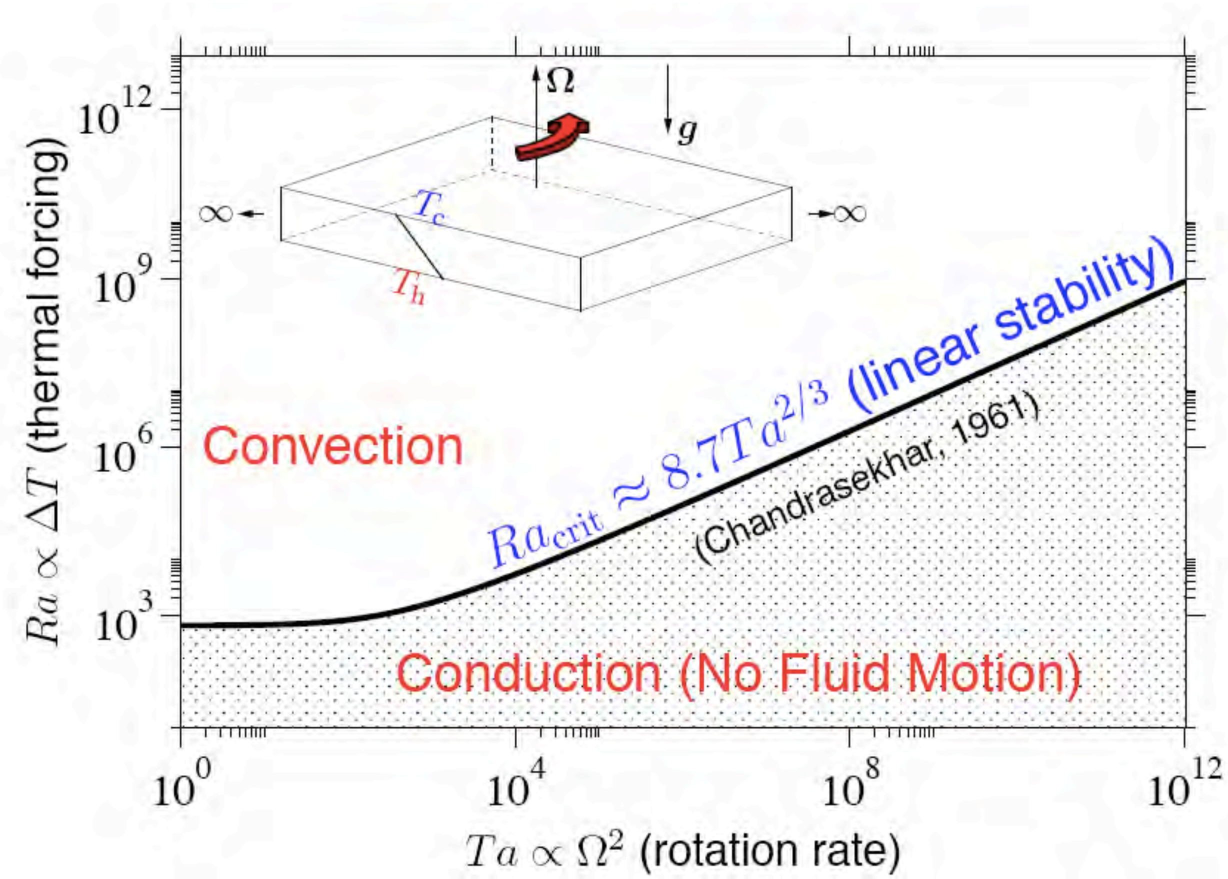

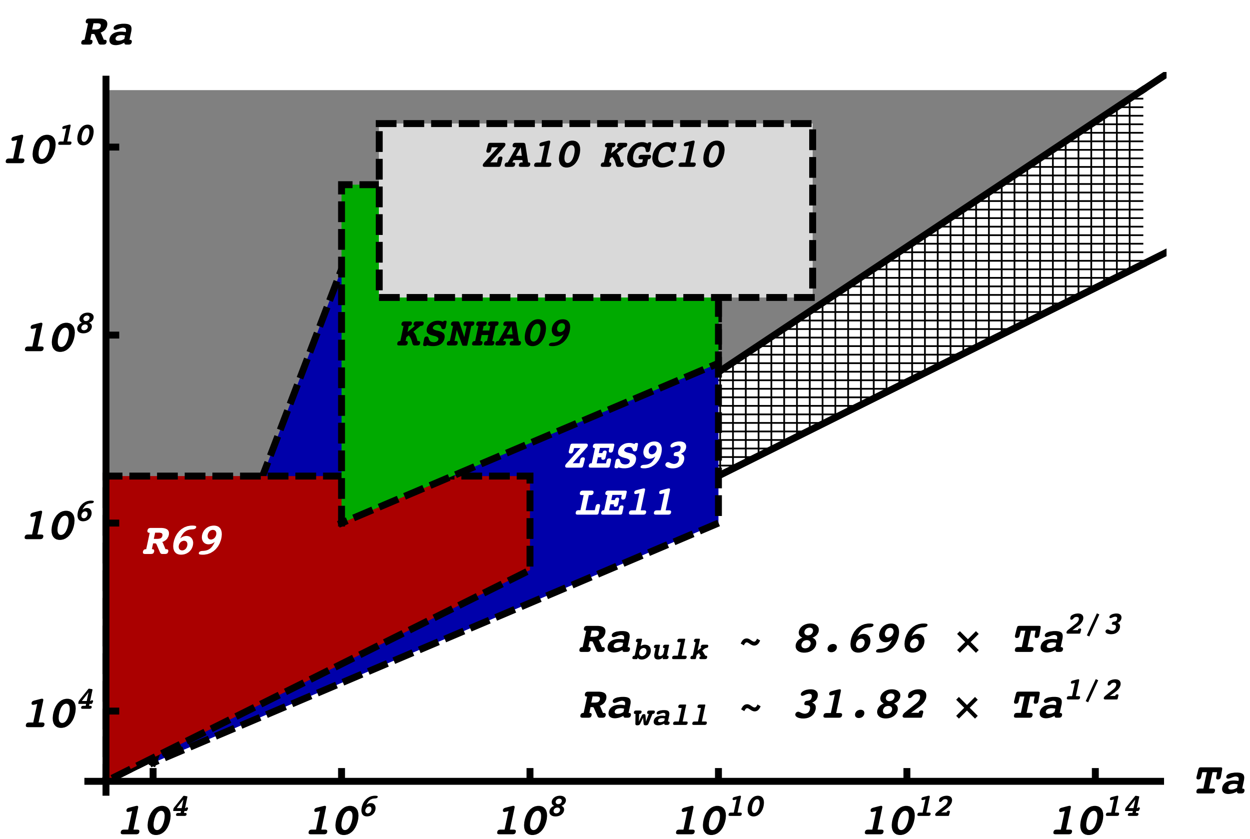

Experiments:

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Laboratory Experiments are limited by engineering and fluid properties

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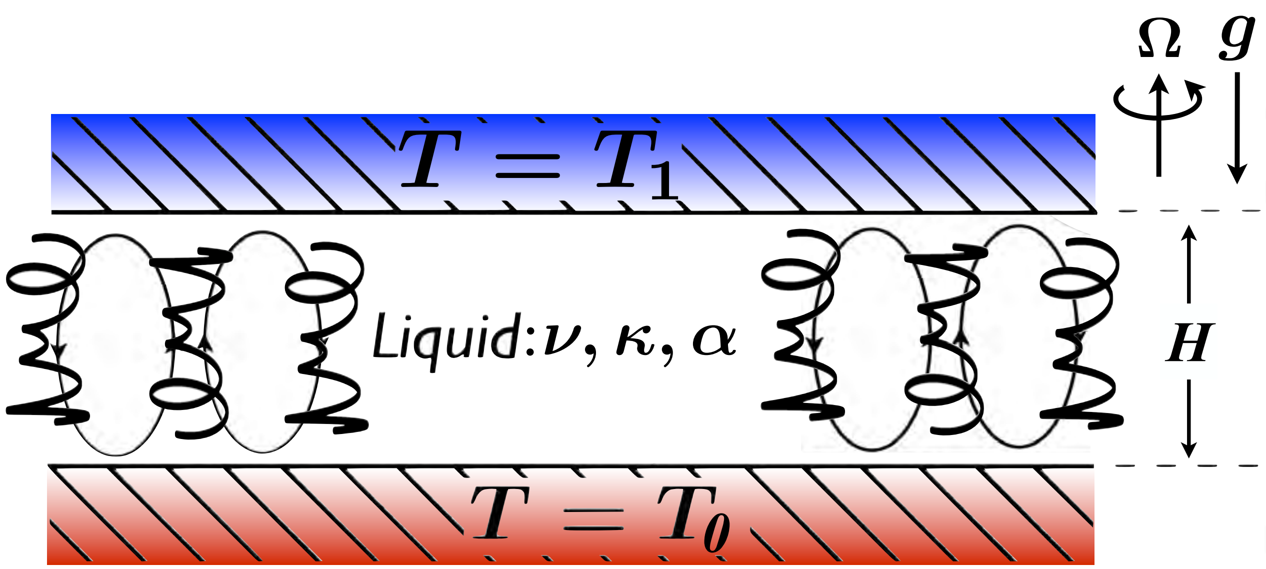

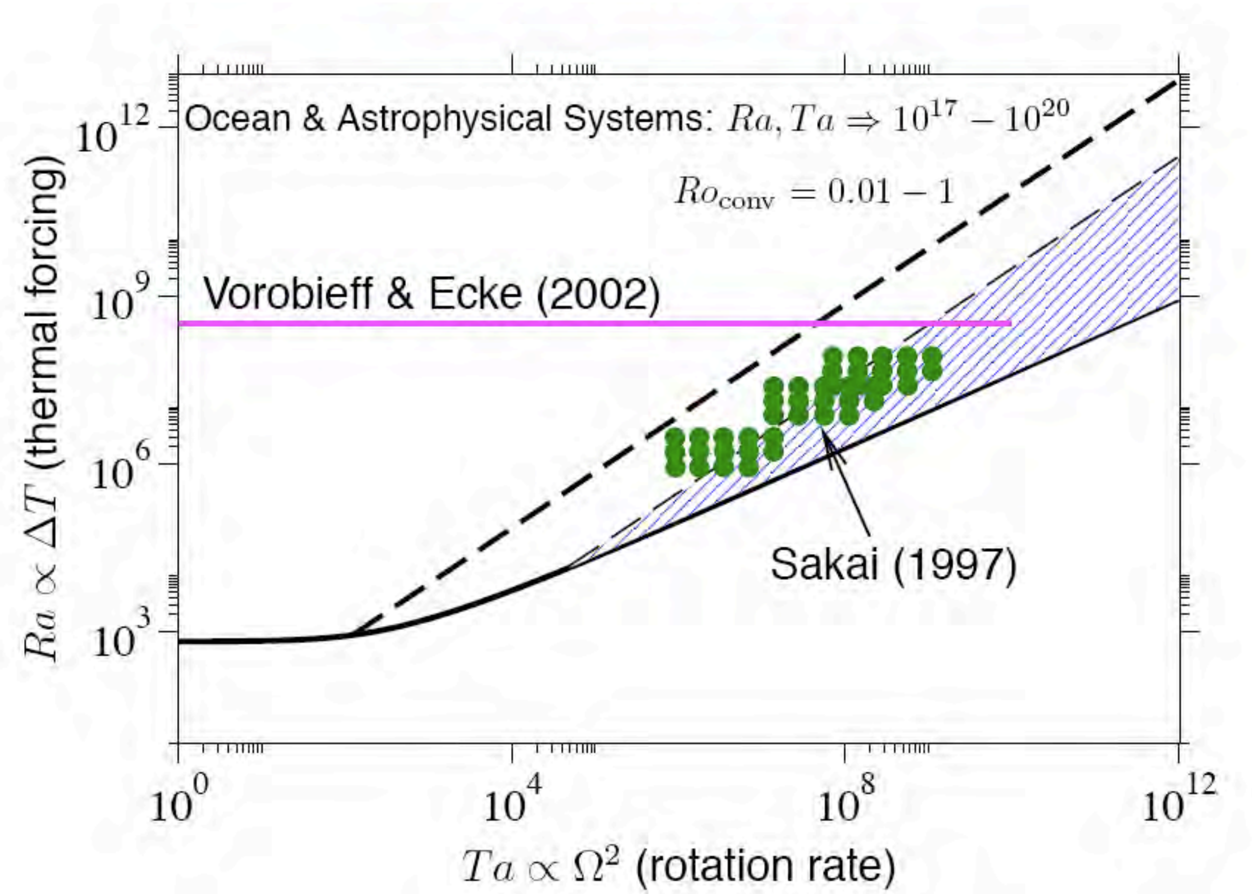

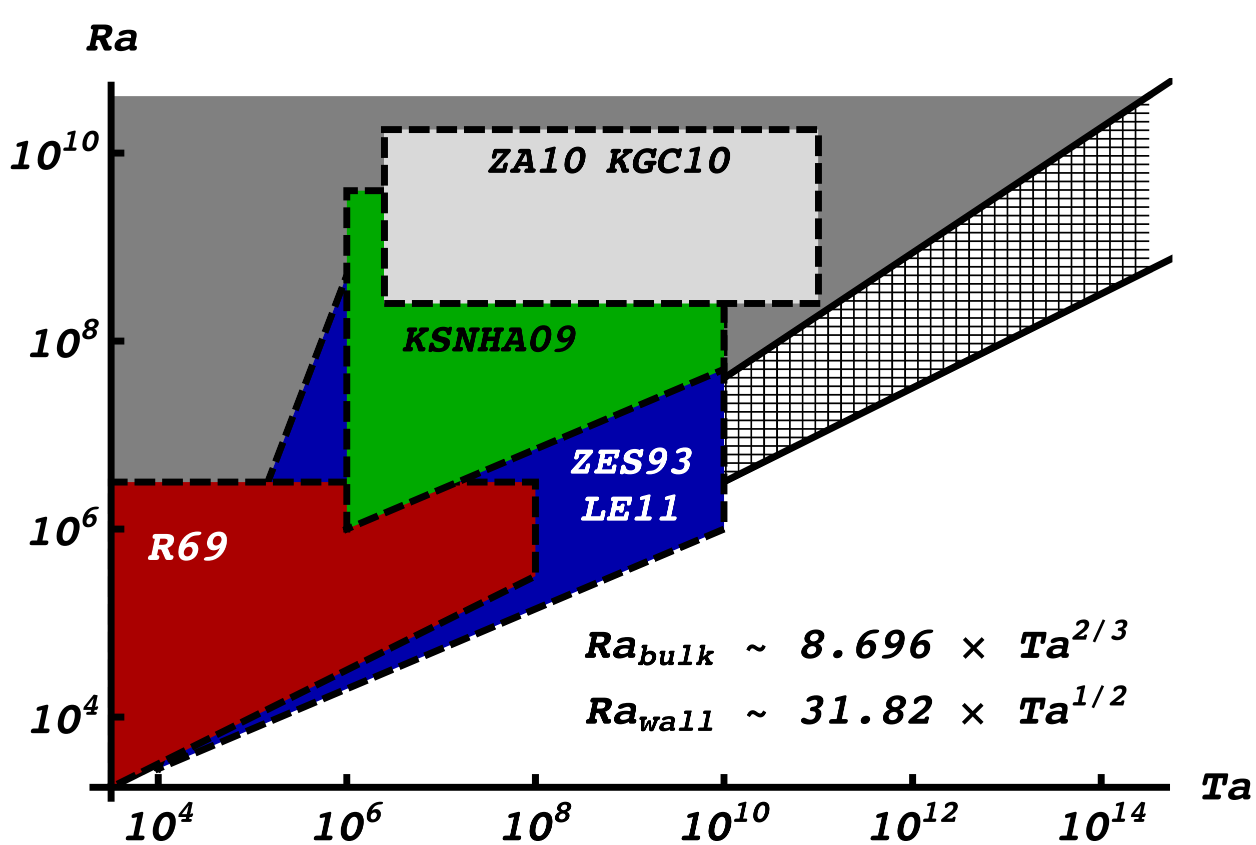

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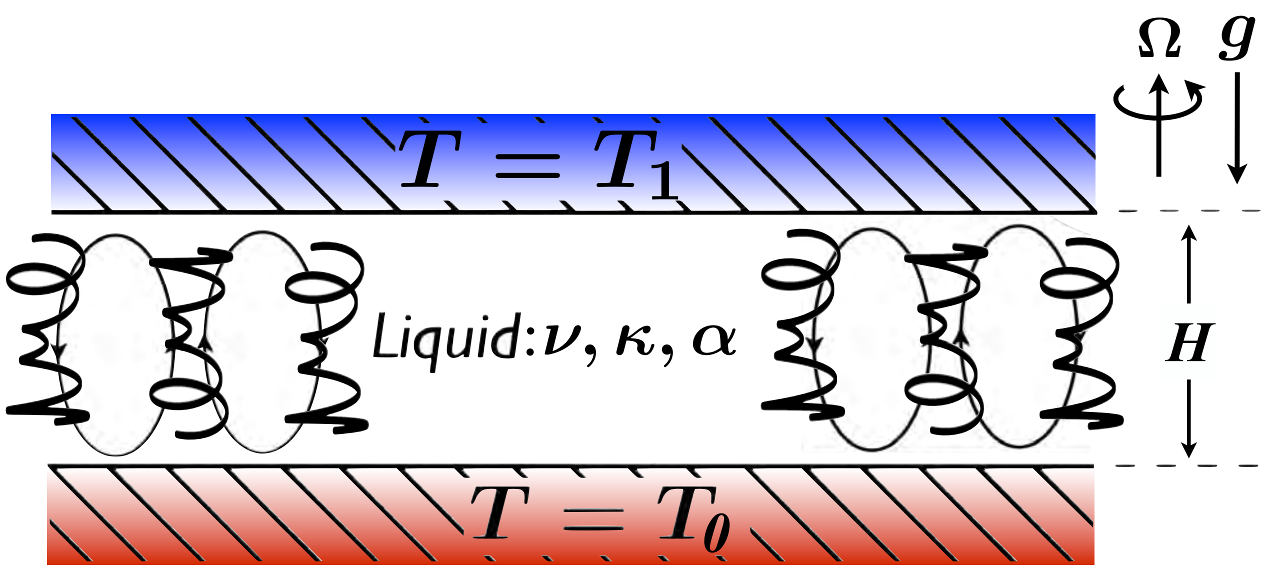
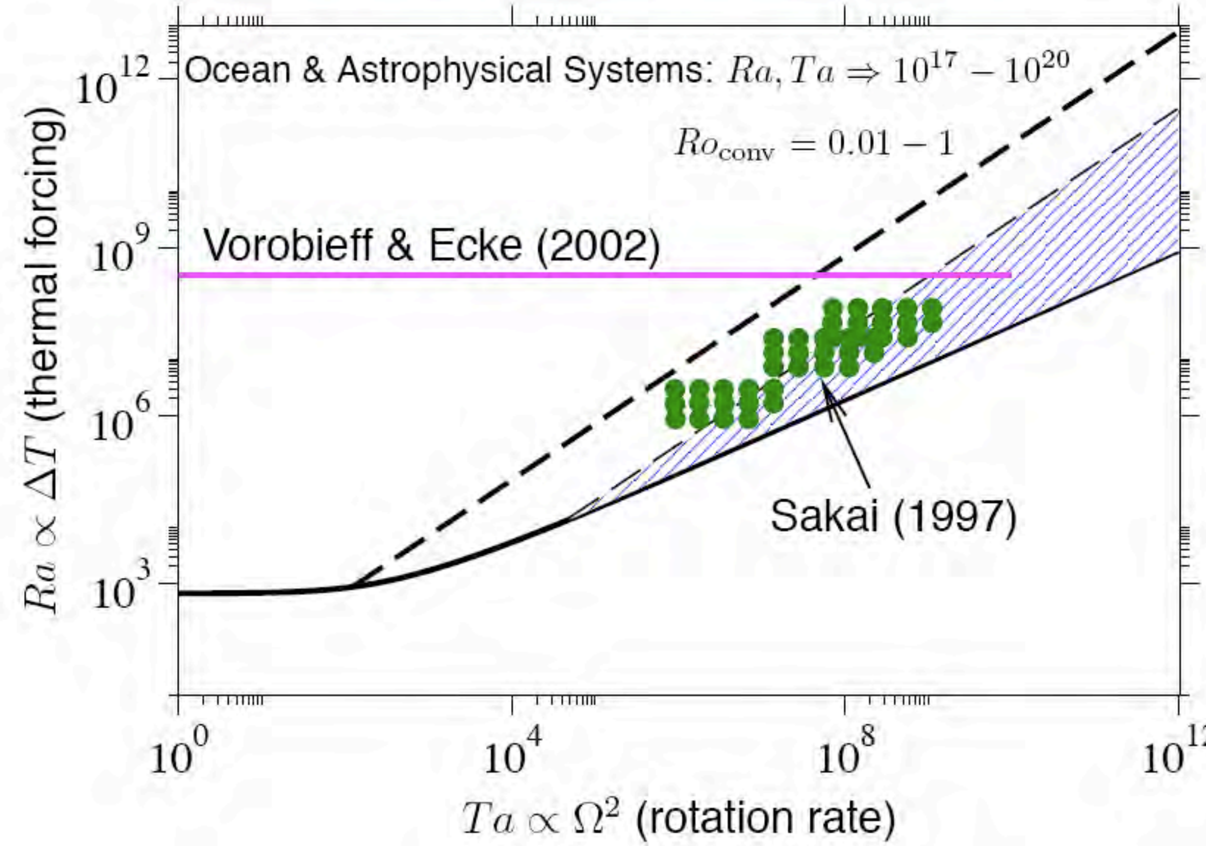
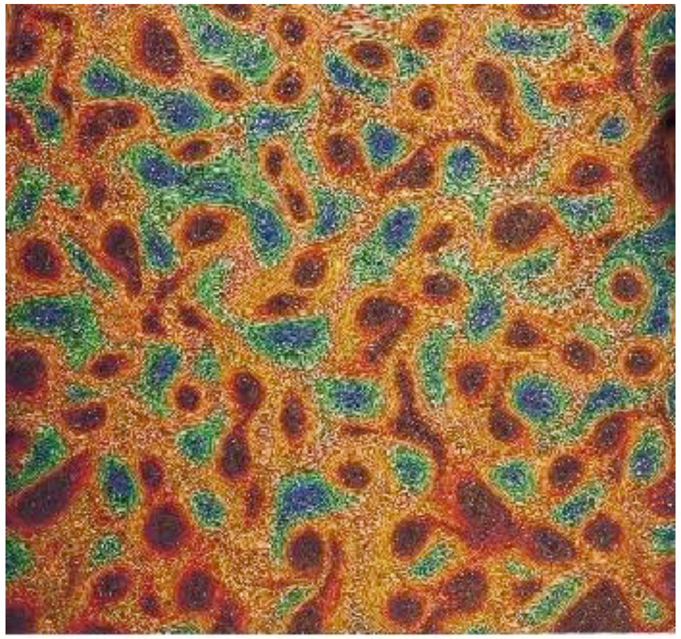
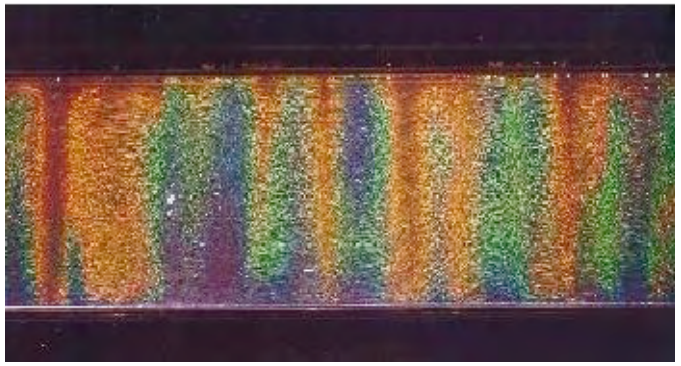




Sakai, JFM 1997: $RaE^{4/3} = 36$, $Ro \approx 0.1$, $\sigma = 7$





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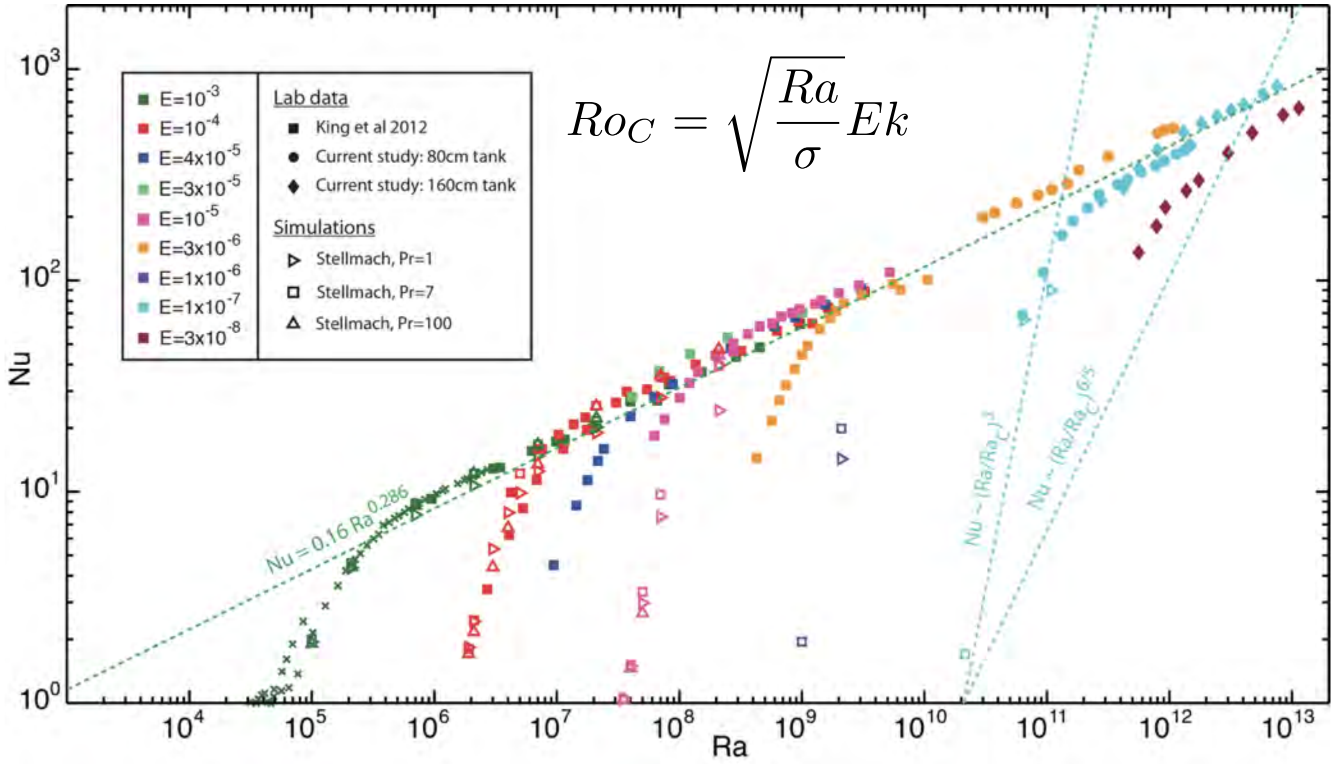
Widely held belief that rotationally constrained motions are strictly columnar

RRBC Results

Parameterization: dependence of global fluid properties on [Re(Ra), Ro, Ek, Pr]

RRBC Results - Heat Transport



UCLA group: courtesy Aurnou & Cheng
King et al Nature 2009

Parameterization: dependence of global fluid properties on [Re(Ra), Ro, Ek, Pr]

- Heat Transport - Nusselt Number

$$Nu - 1 \propto \sigma^{\alpha} (Ra/Ra_c)^{\beta}$$

- Flow Morphology
  - Pathway to geostrophic turbulence at low Ro

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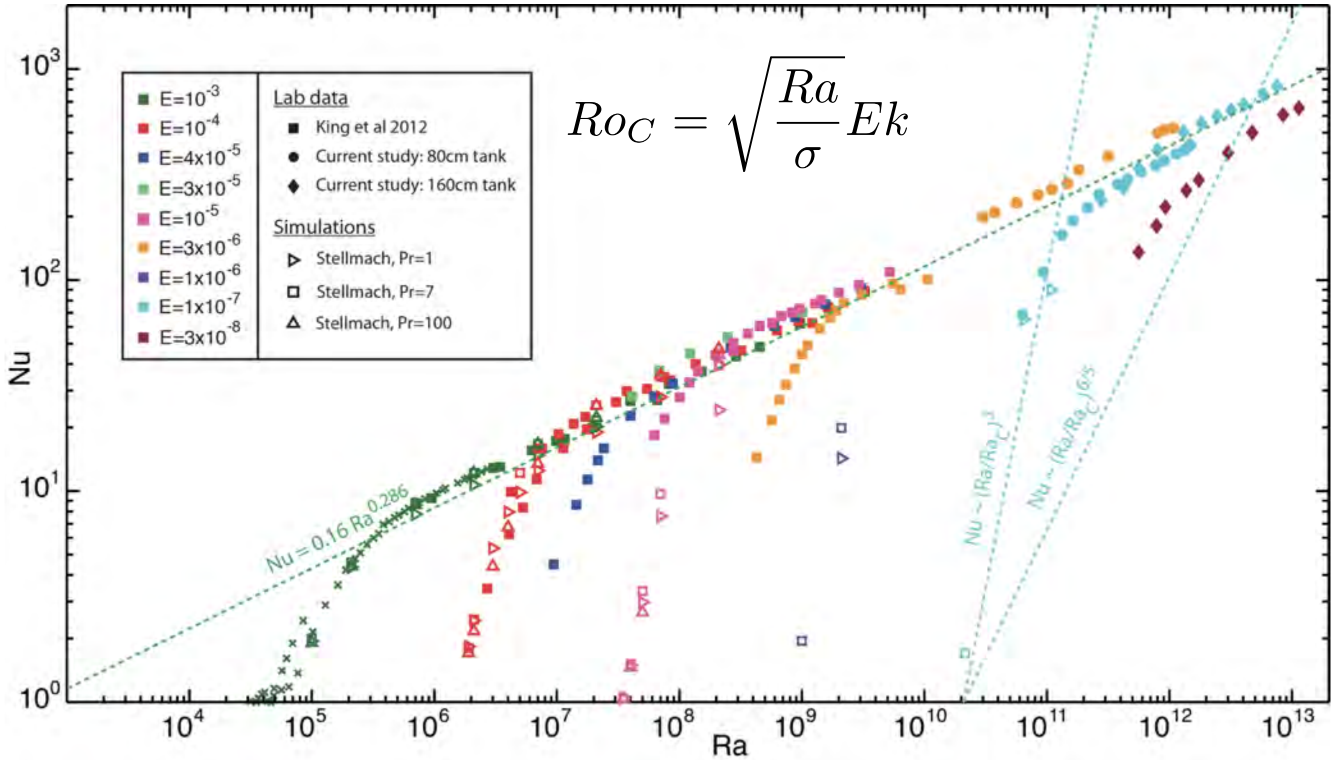

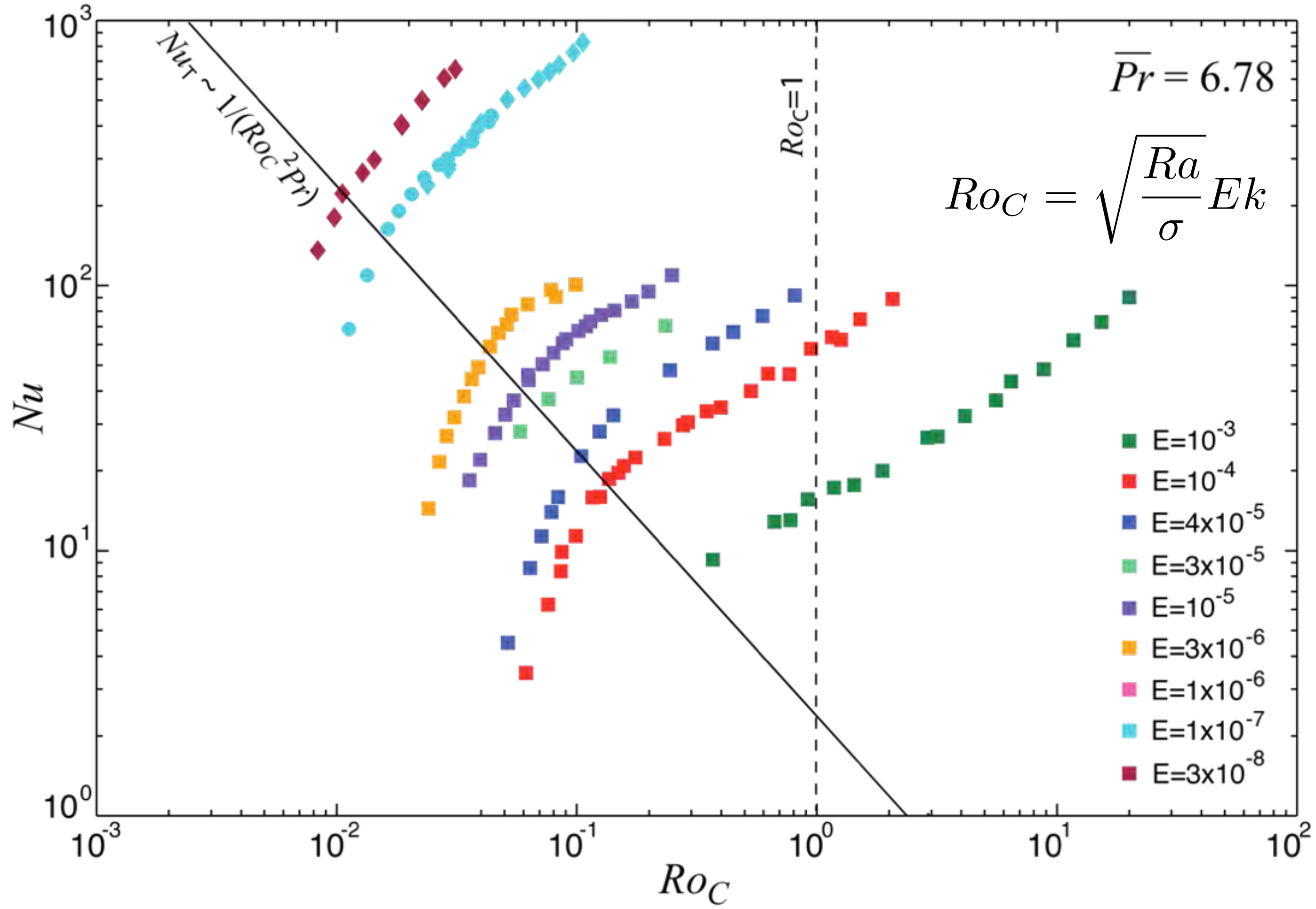

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  - Low Ro transition to non-rotating scaling law
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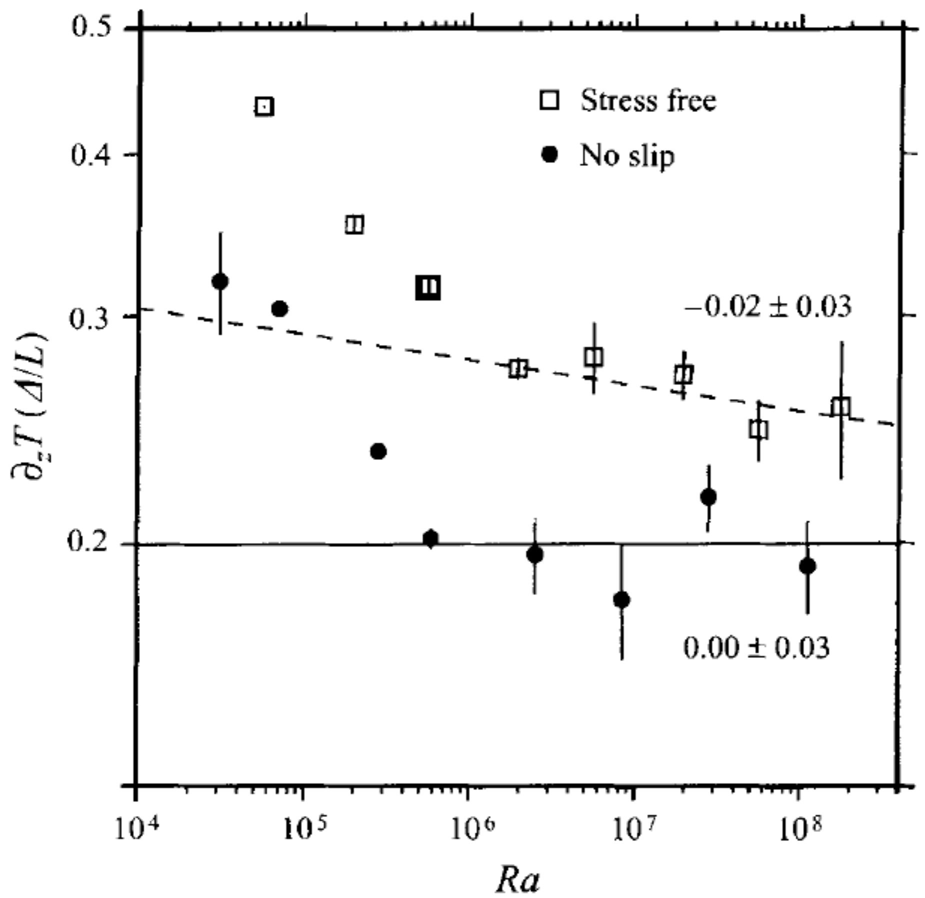
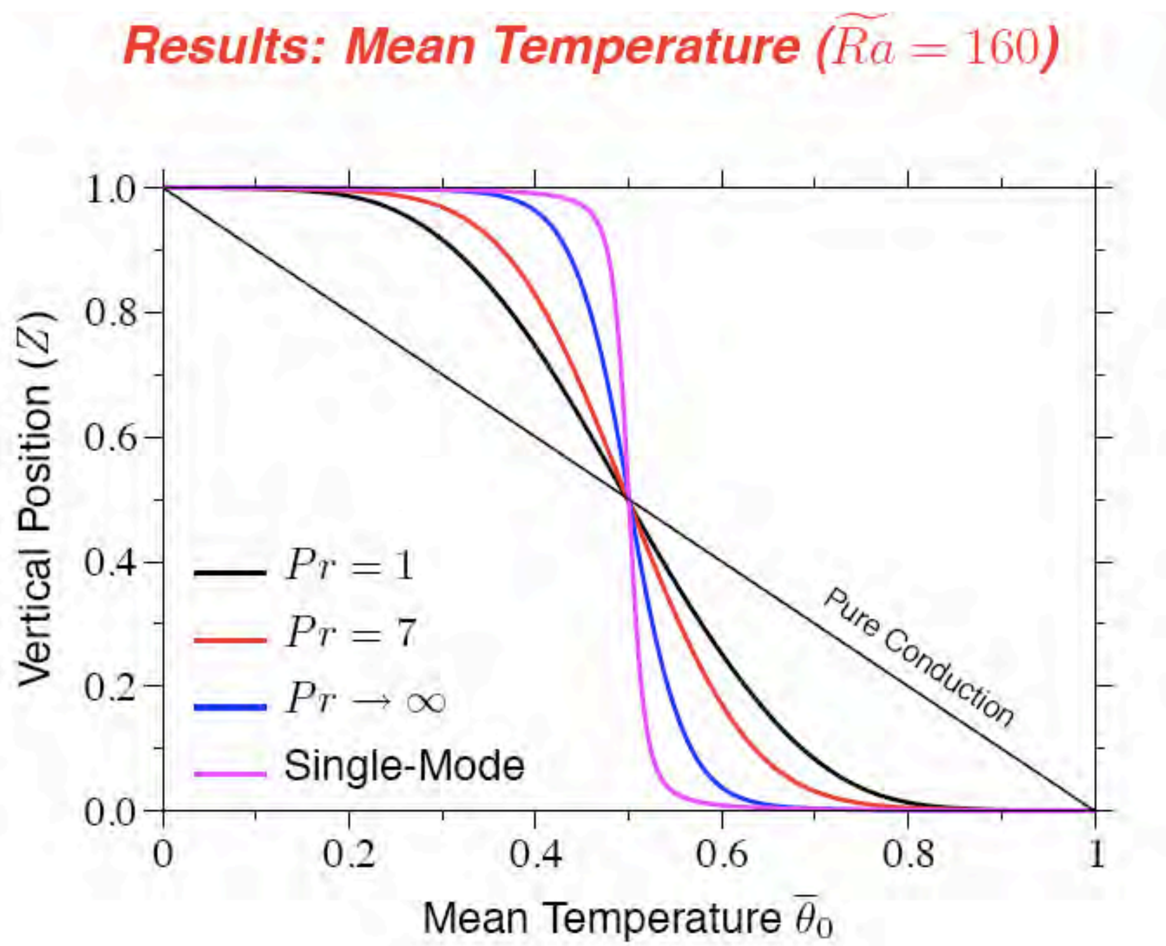
RRBC Results - Lateral Mixing



J. et al JFM 1996



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- Mixing
  - saturation of mean temp. gradient

$$\partial_z \overline{T}_{mid} \propto \sigma^{\gamma}(Ra_T/Ra_c)^{\delta}$$

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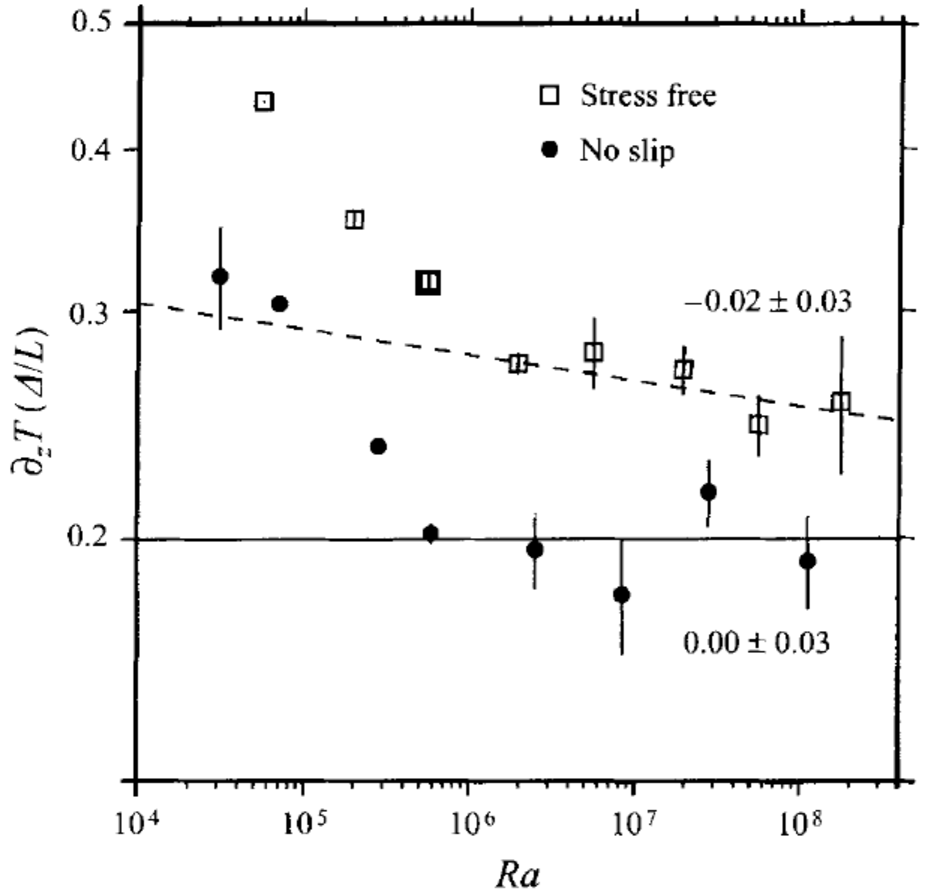

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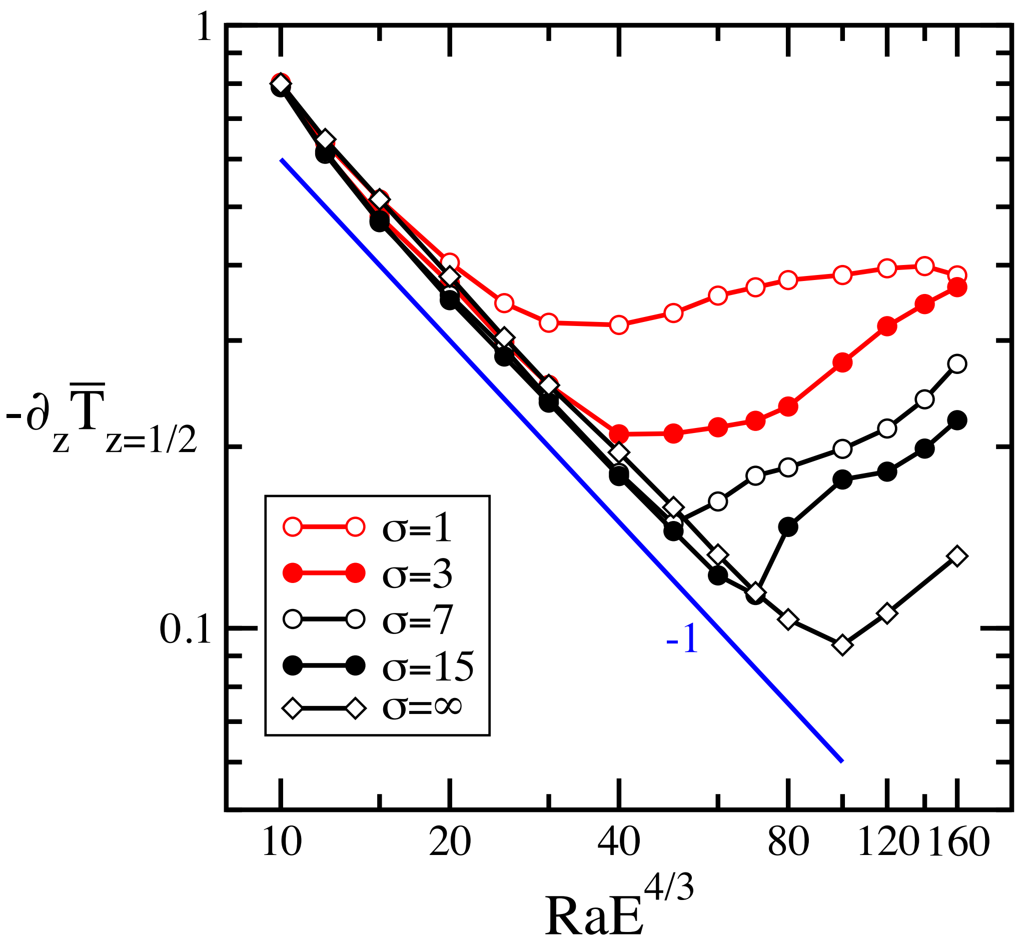

Sprague et al JFM 2006

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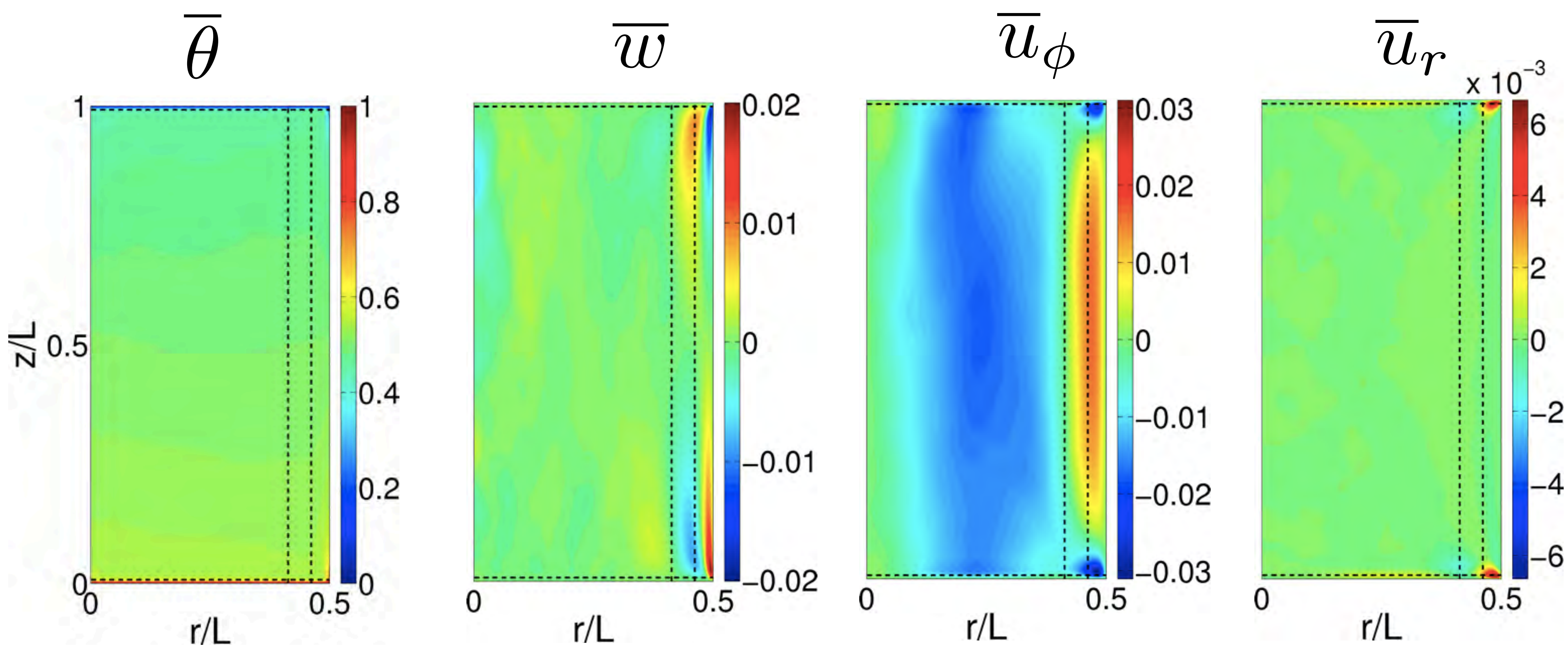
RRBC Results - Large Scale Flow Generation



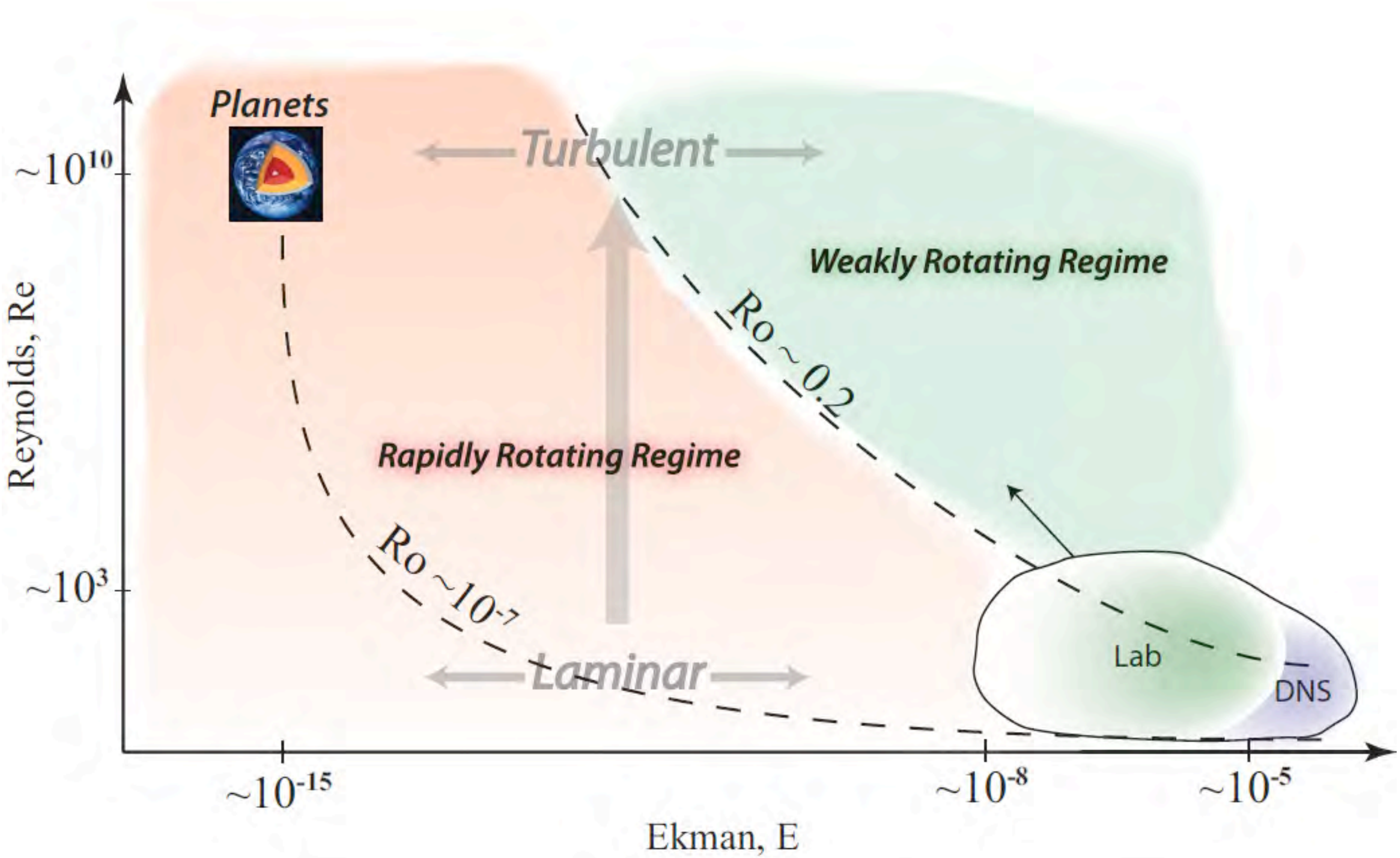
Ra=10^9, Ro=0.36, σ = 7, Kunnen. et al JFM 2011

Convection appears to drive large scale (barotropic) dynamics.

Low Rossby Number Computational Challenge

- Fast waves + geostrophically balanced eddies limit DNS/Lab investigations

$$\partial_t \boldsymbol{u} + Ro^{-1}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0$$



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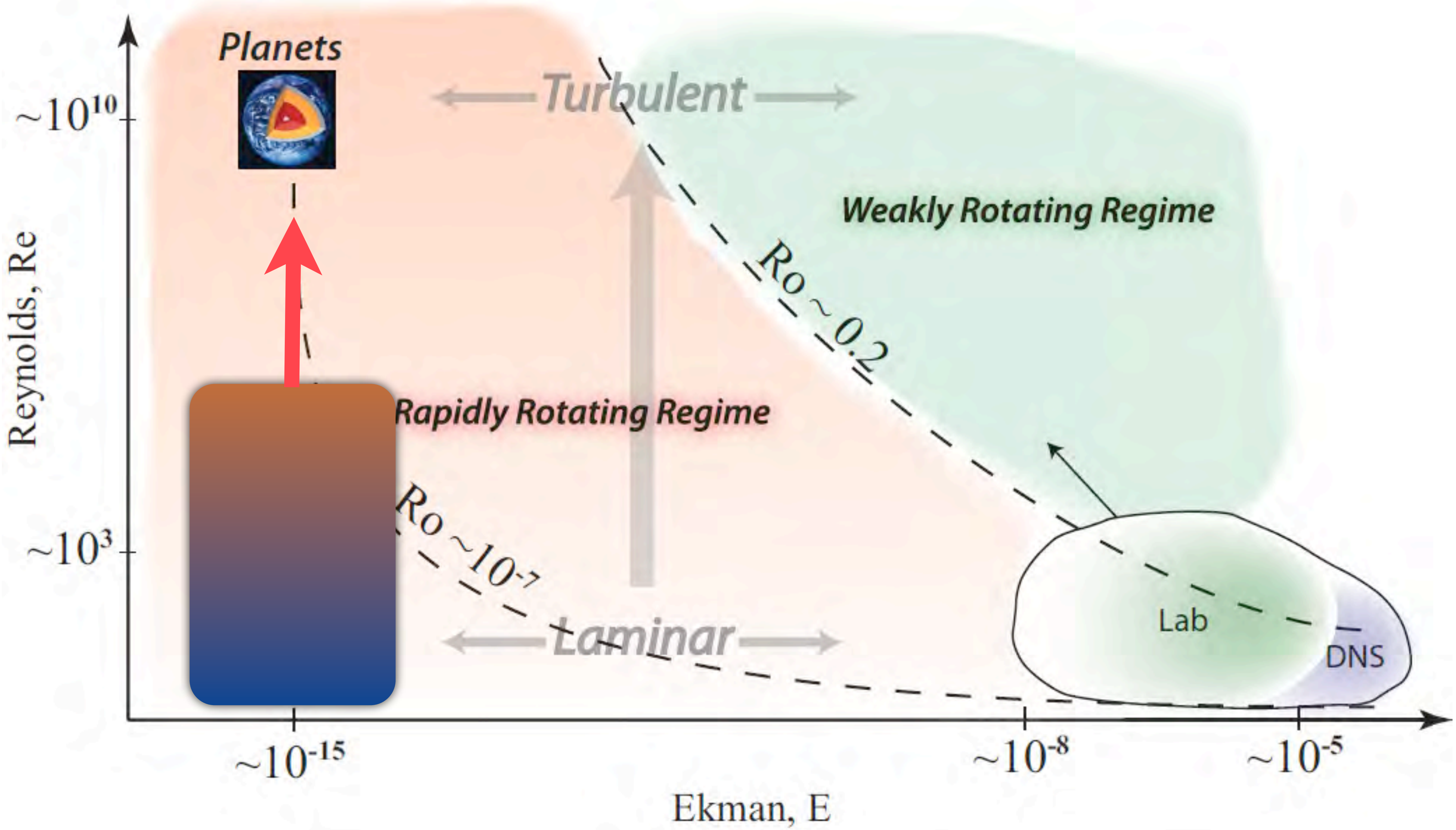
- Existence of reduced PDE models that filter fast waves and automatically enforce geostrophic balance?



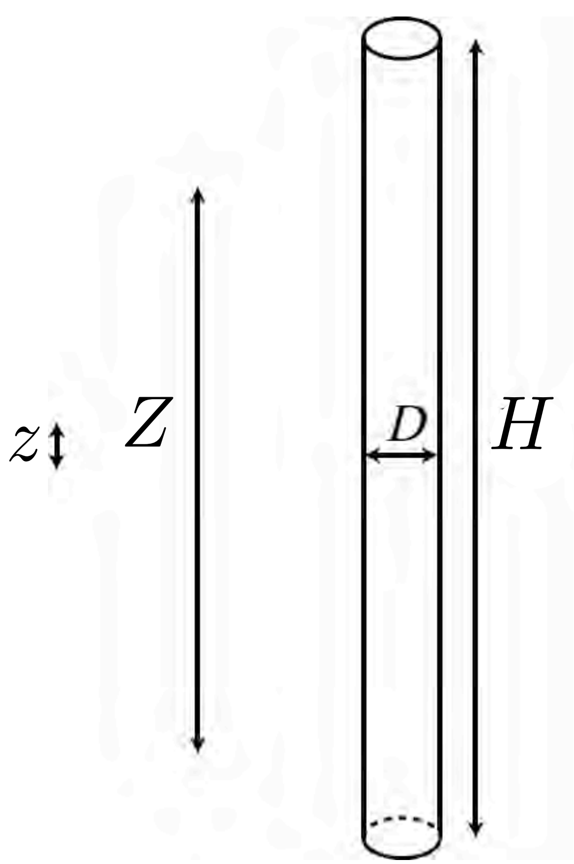
Multi-Scale Asymptotics to the Rescue

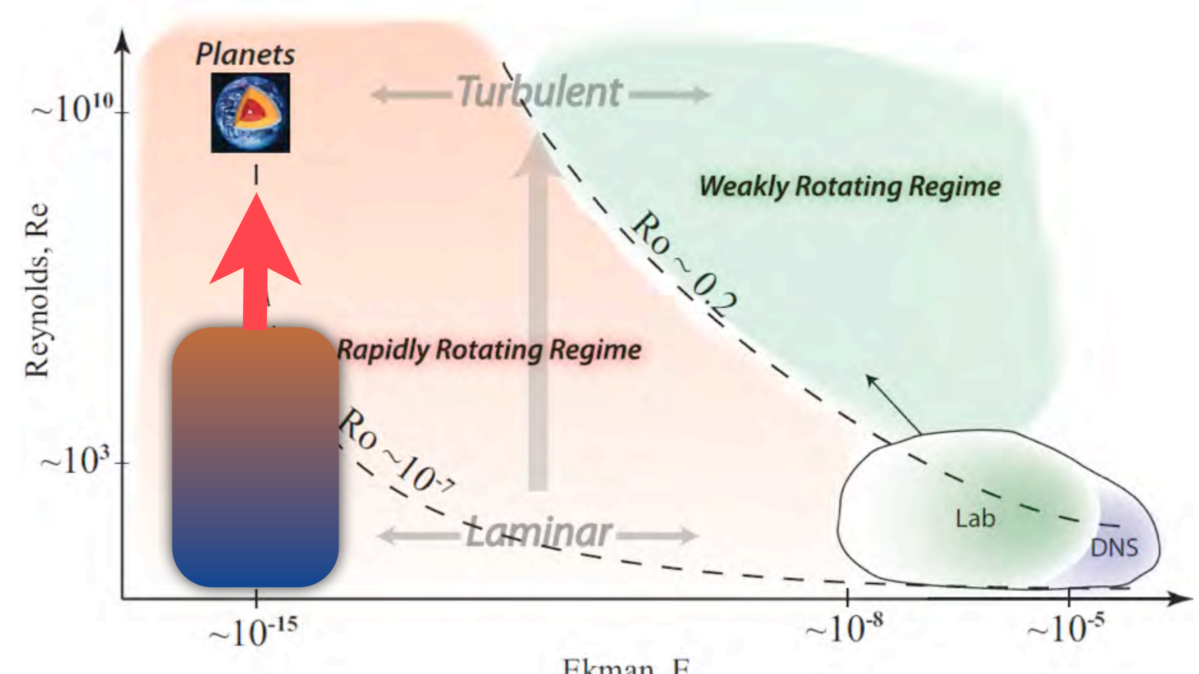
- Exploit small parameters asymptotically (Ro, Ek): $\boldsymbol{u} = \boldsymbol{u}_0 + Ro\,\boldsymbol{u}_1 + \cdots$
- Leading order balance, geostrophic approx'n (fast inertial waves filtered; Embid & Majda GAFD '98)

$$\left.\begin{aligned} Ro^{-1}\left(\widehat{\boldsymbol{z}} \times \boldsymbol{u} + \nabla p\right) &\approx 0 \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}\right\} \Longrightarrow \quad \nabla_\perp \cdot \boldsymbol{u}_\perp \approx 0 \qquad \partial_z \left(\boldsymbol{u}_\perp, w, p\right) \approx 0$$

T-P constraint

- Diagnostic solution: $\boldsymbol{u} \approx -\nabla \times \psi \widehat{\boldsymbol{z}} + w\widehat{z}, \qquad p = \psi \qquad \zeta = \nabla_\perp^2 \psi$
- Quasigeostrophic perturbation theory, solvability: $\widehat{\boldsymbol{z}} \cdot \nabla \times\,, \quad \widehat{\boldsymbol{z}} \cdot \quad \Leftrightarrow \quad \overline{f} = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} f\, dz$
- Reinterpret Taylor-Proudman theory (MSA; KJ, Knobloch, Milliff & Werne, JFM 2006)





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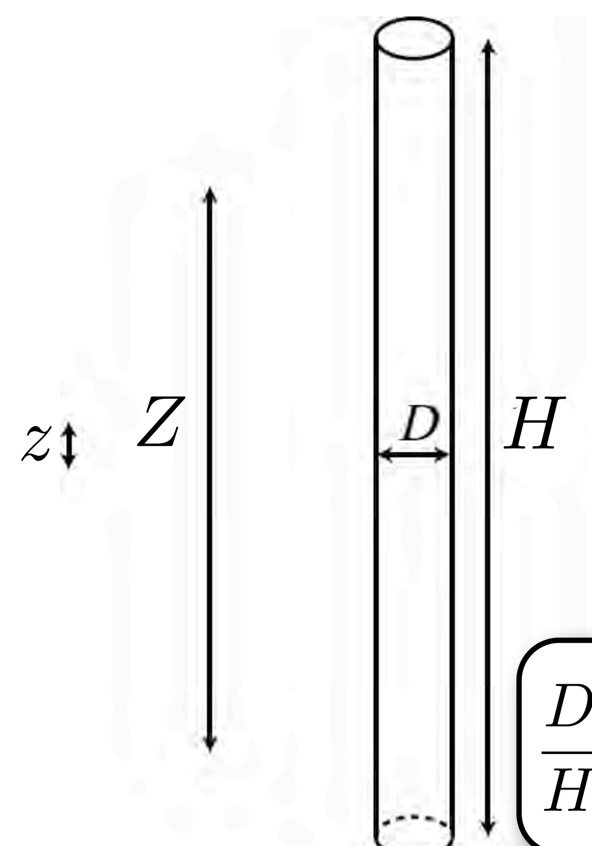
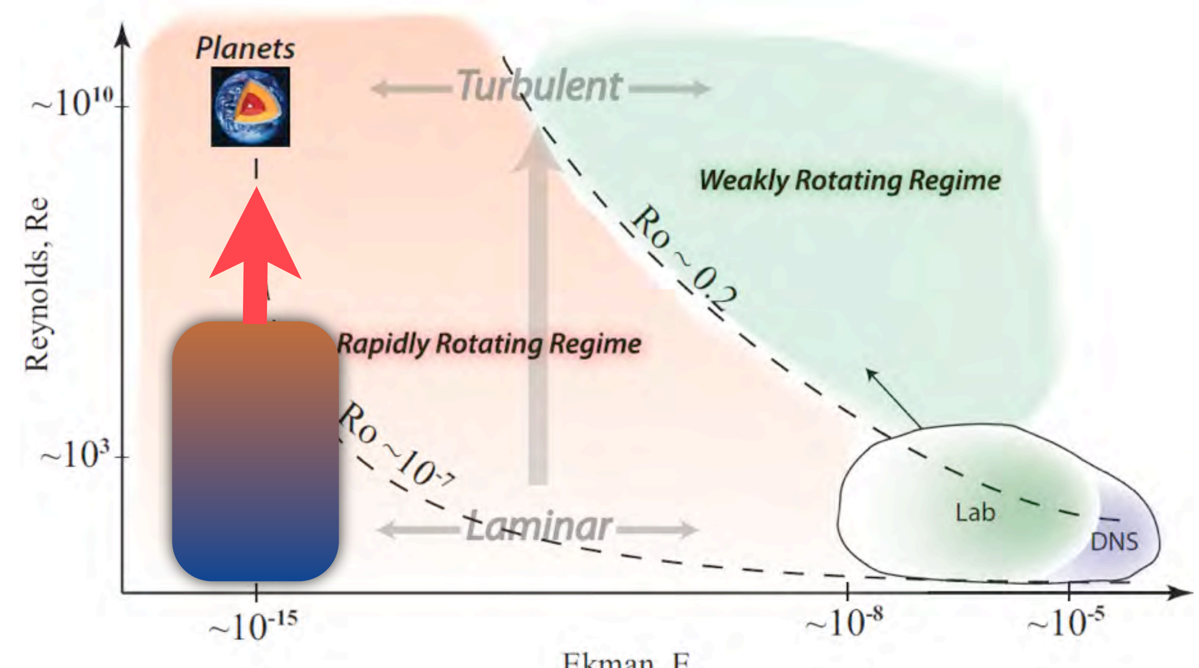


$$\partial_z \longrightarrow \not{\partial}_z + Ro\,\partial_Z$$

$$\frac{D}{H} \sim Ro$$



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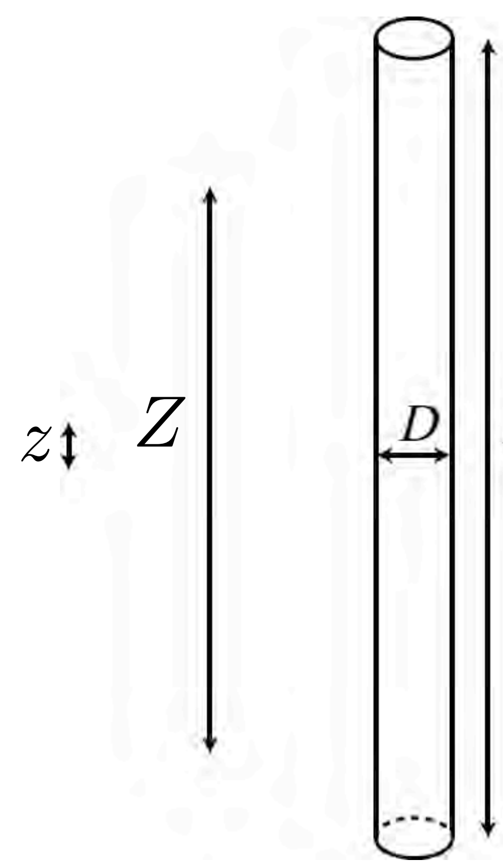

$$\partial_t \zeta + \boldsymbol{u}_\perp \cdot \nabla_\perp \zeta - \partial_Z w = \nabla_\perp^2 \zeta + \widehat{\boldsymbol{z}} \cdot \nabla \times \overline{\boldsymbol{S}}$$

$$\partial_z \rightarrow \cancel{\partial_z} + Ro\,\partial_Z$$

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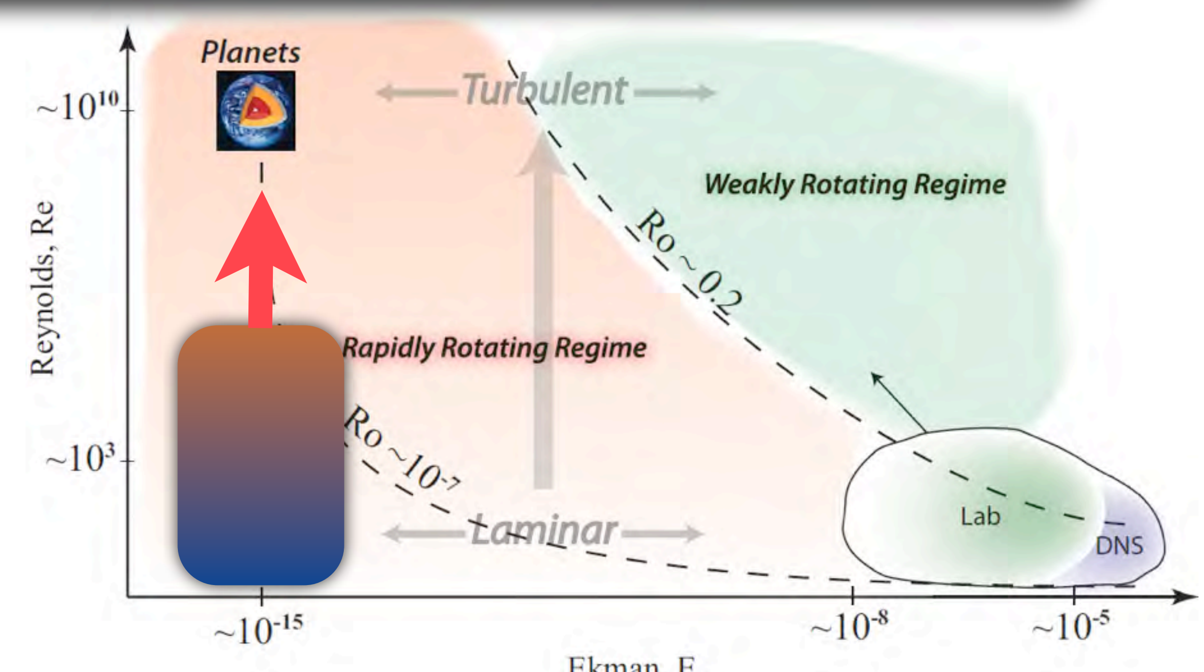
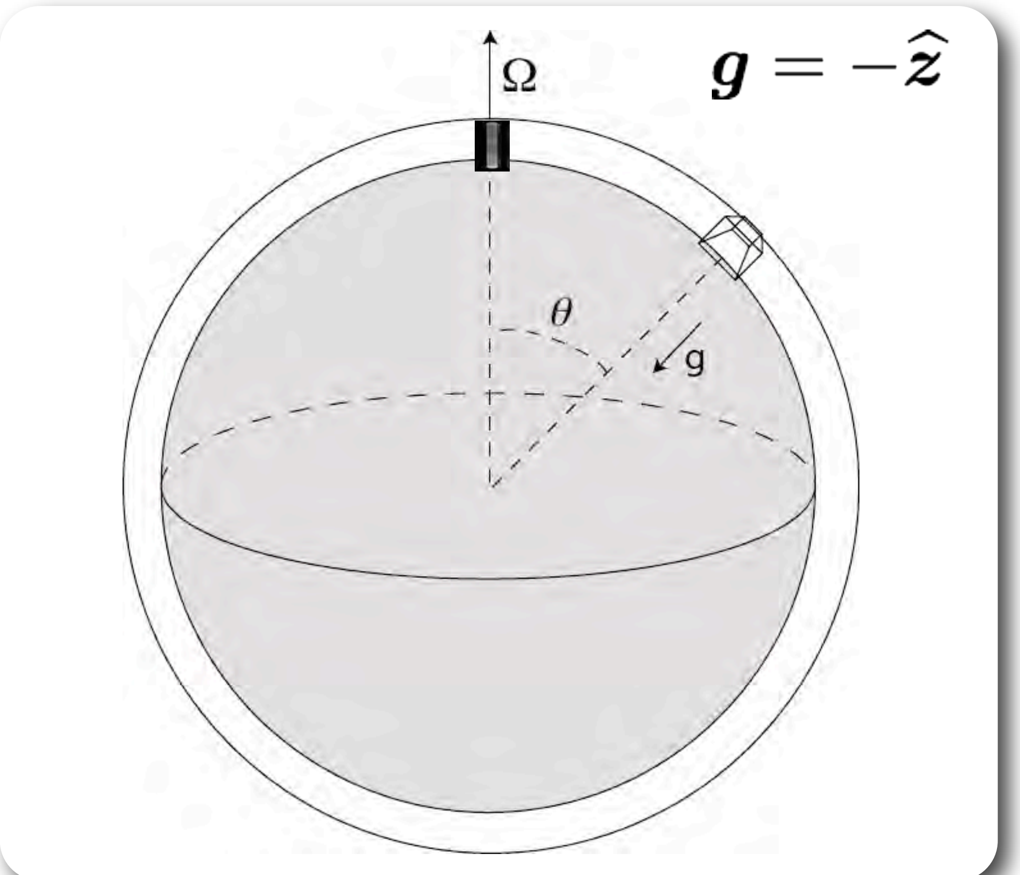
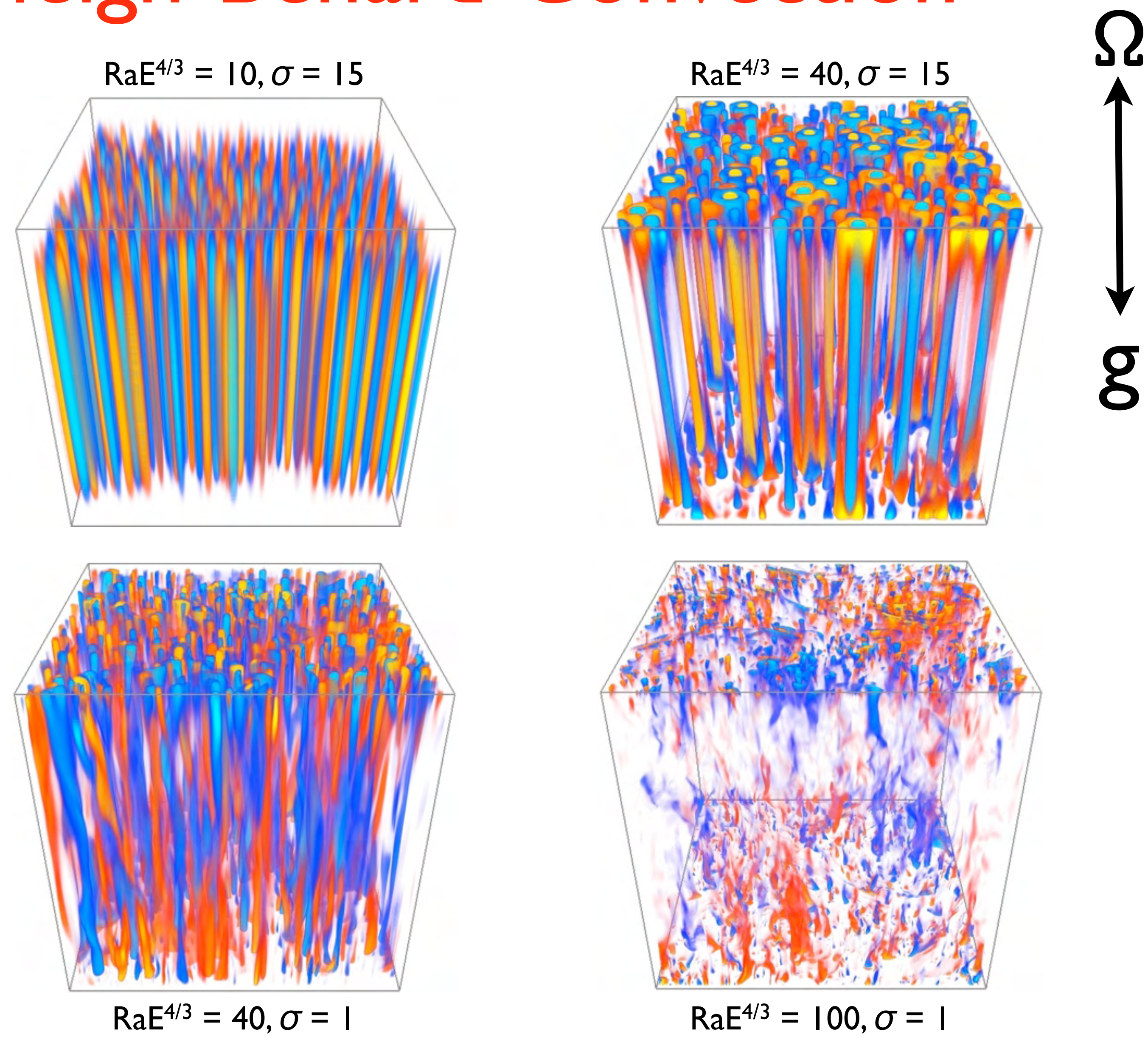


Quasigeostrophic Rayleigh-Bénard Convection





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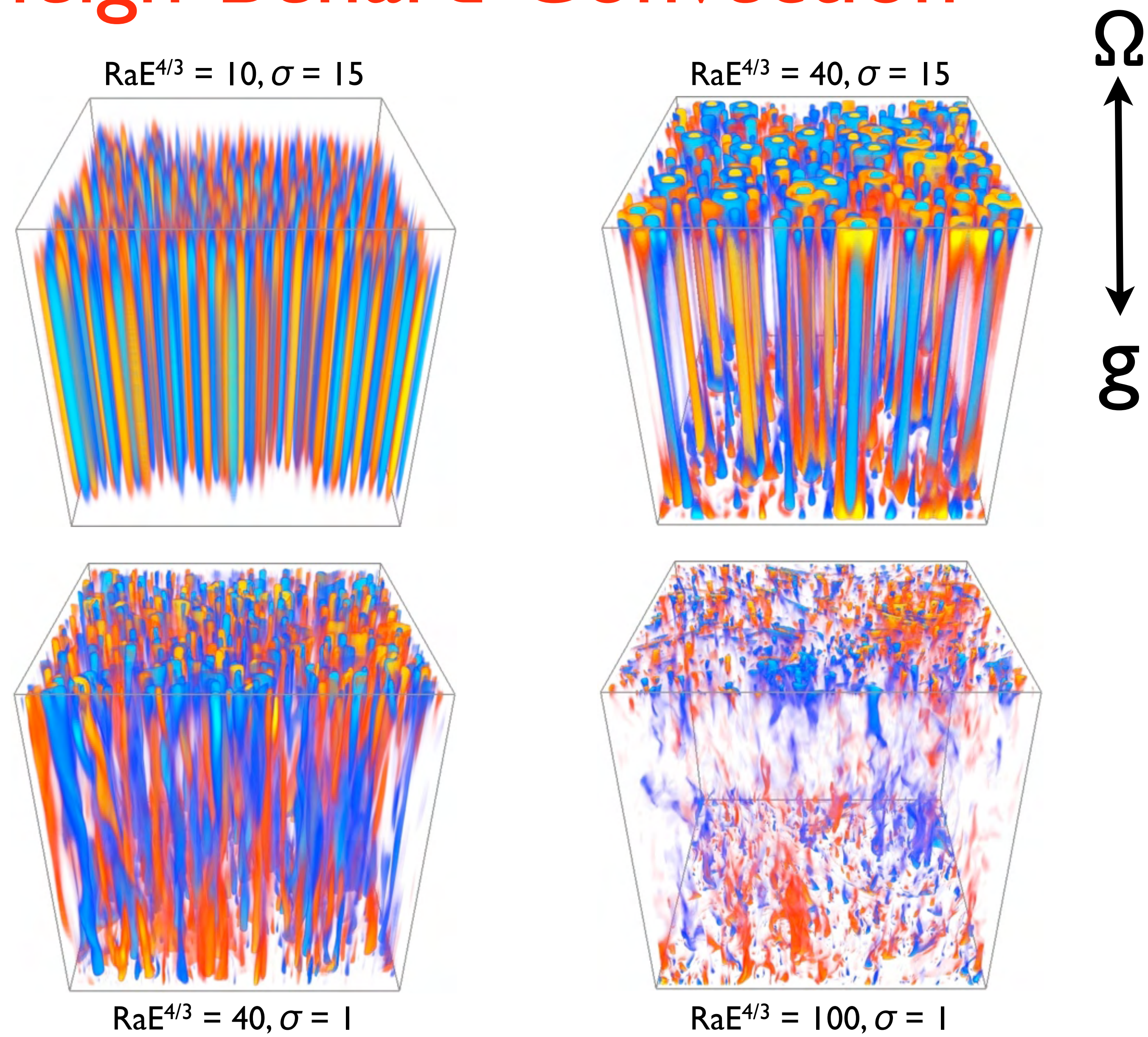
J. et al JFM 2006, GAFD '12



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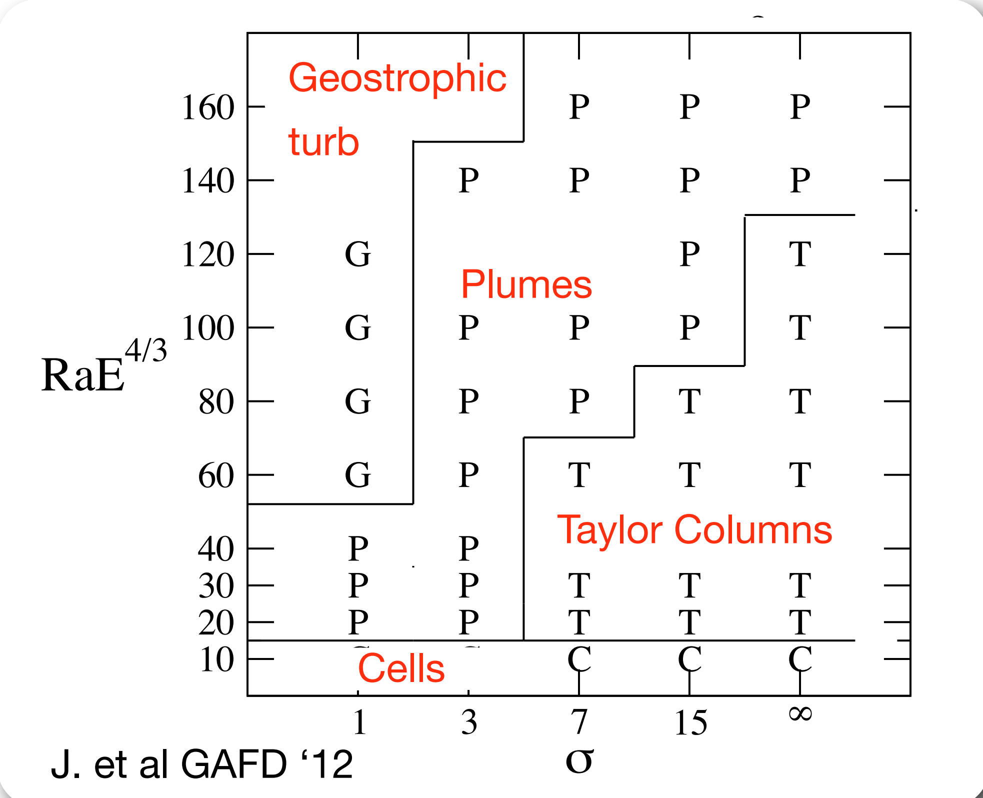

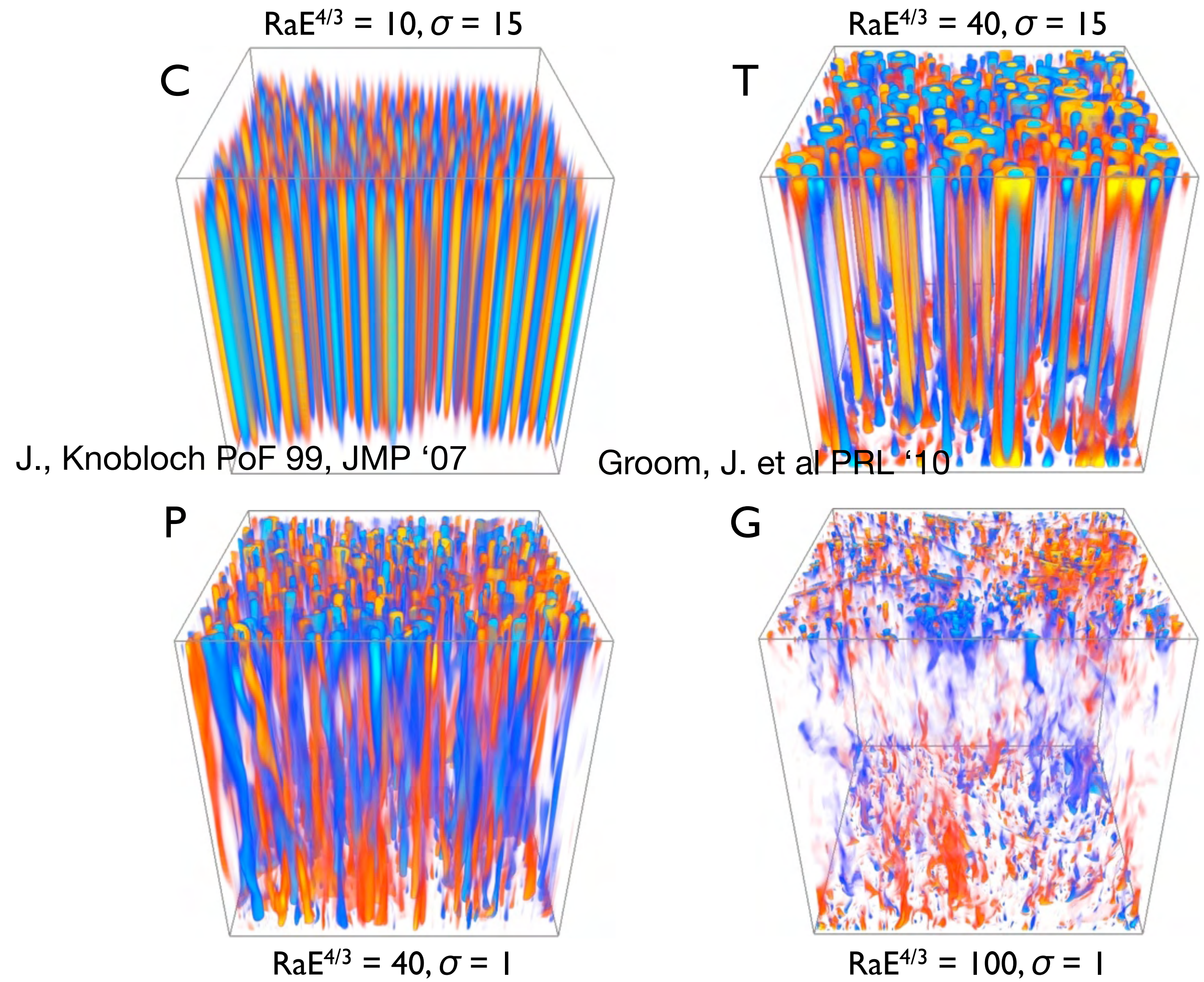

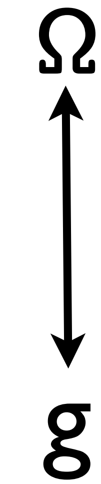

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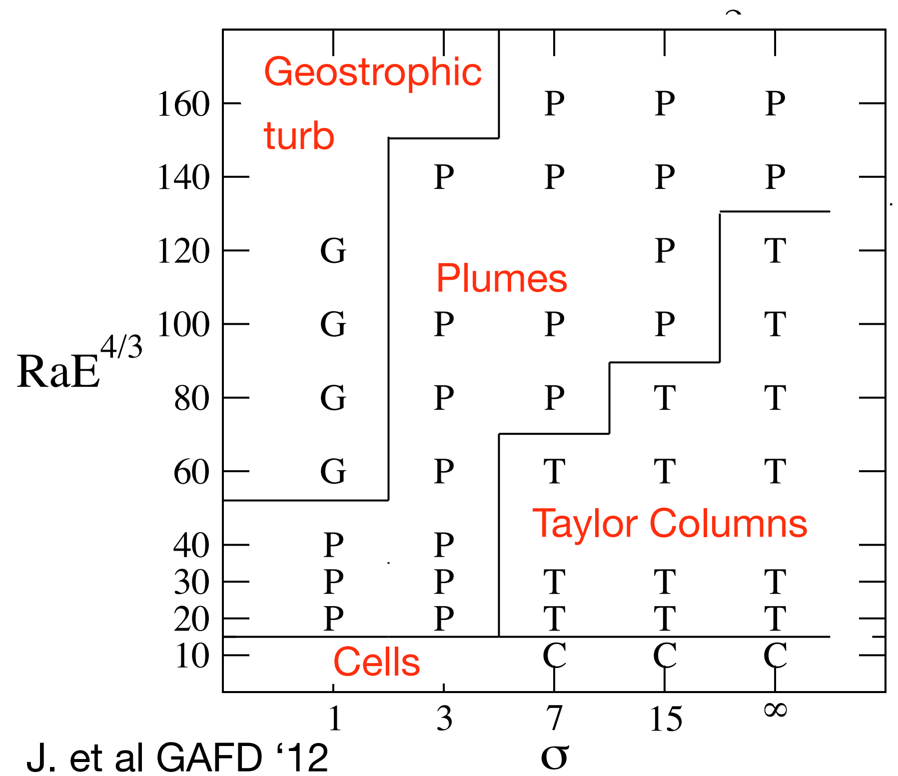
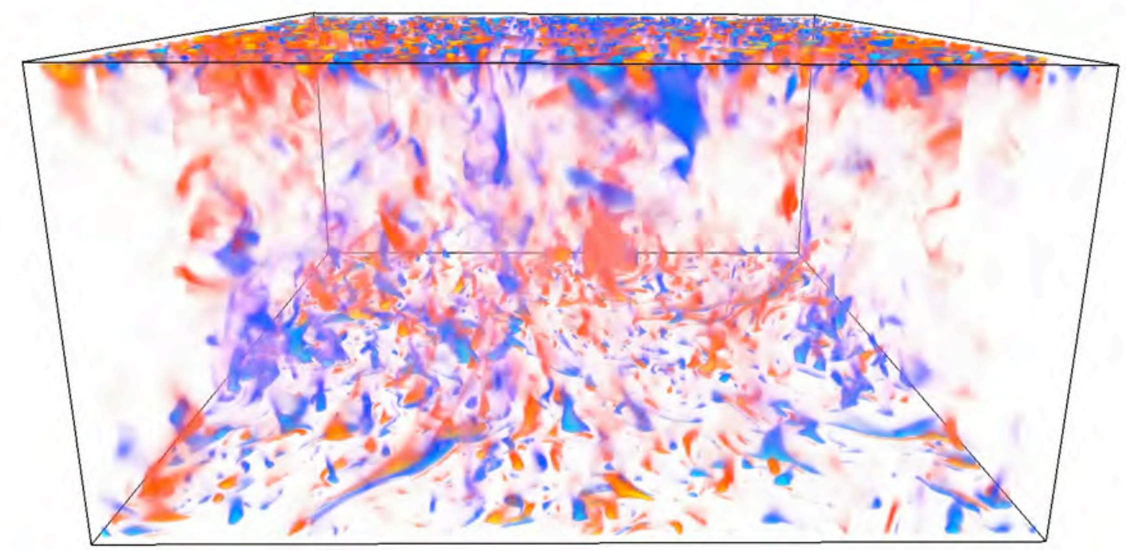
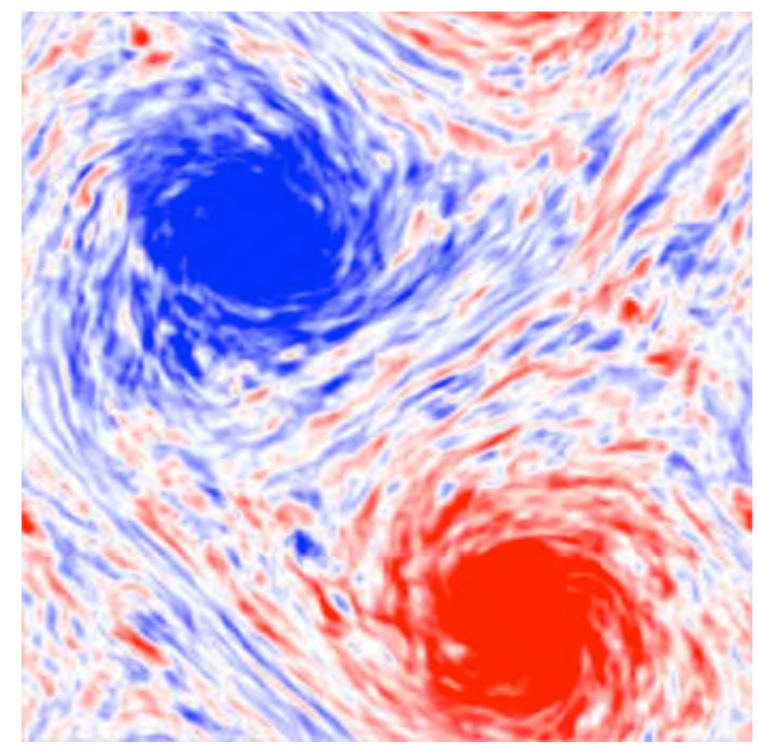


J. et al GAFD '12

RaE$^{4/3}$=100, Pr=1



J. et al GAFD '12; Rubio, J., Weiss submitted '13



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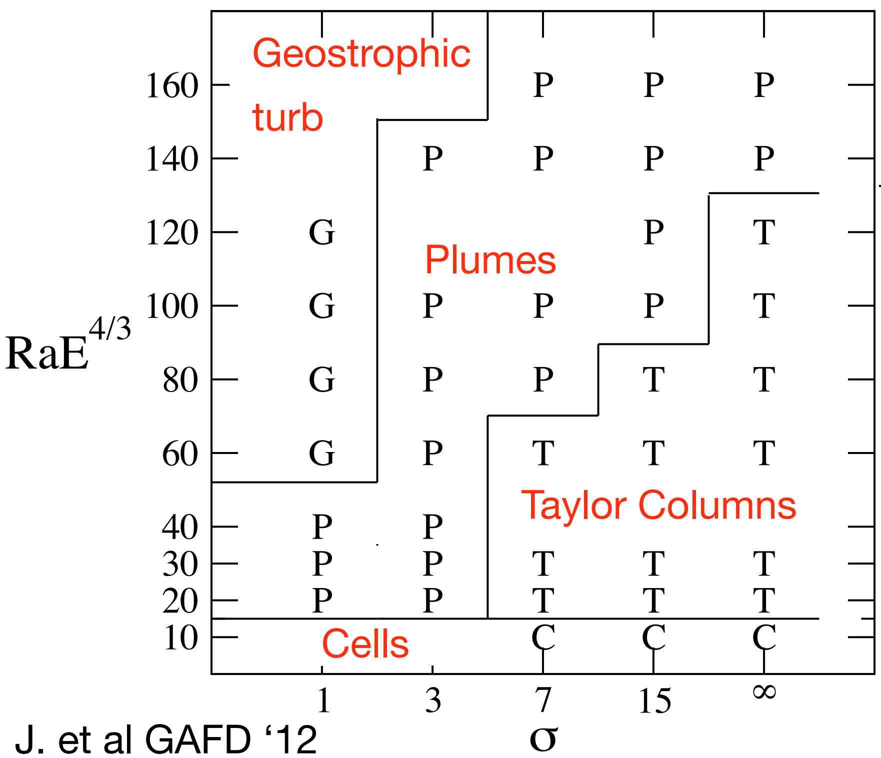

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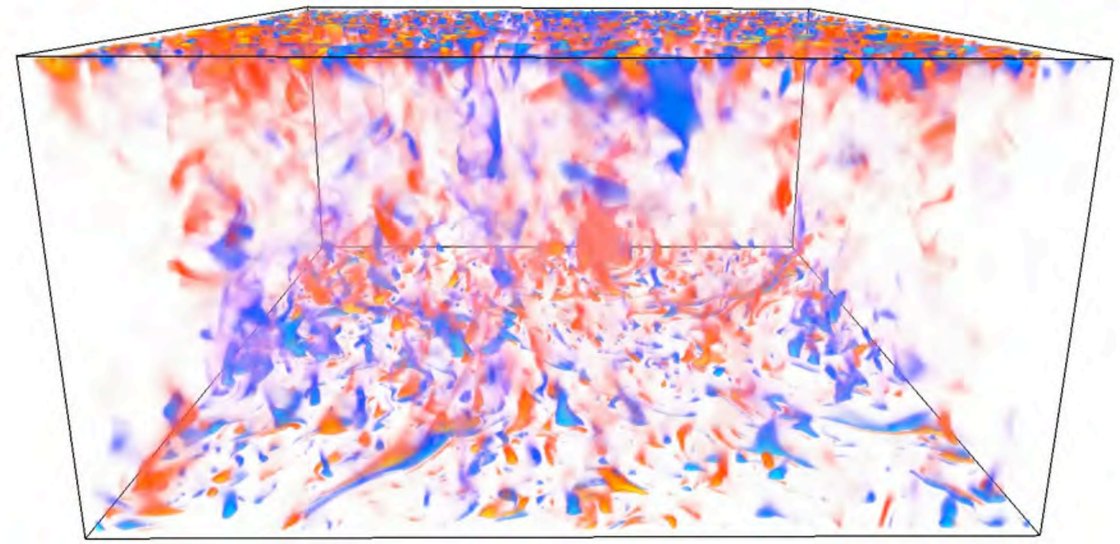

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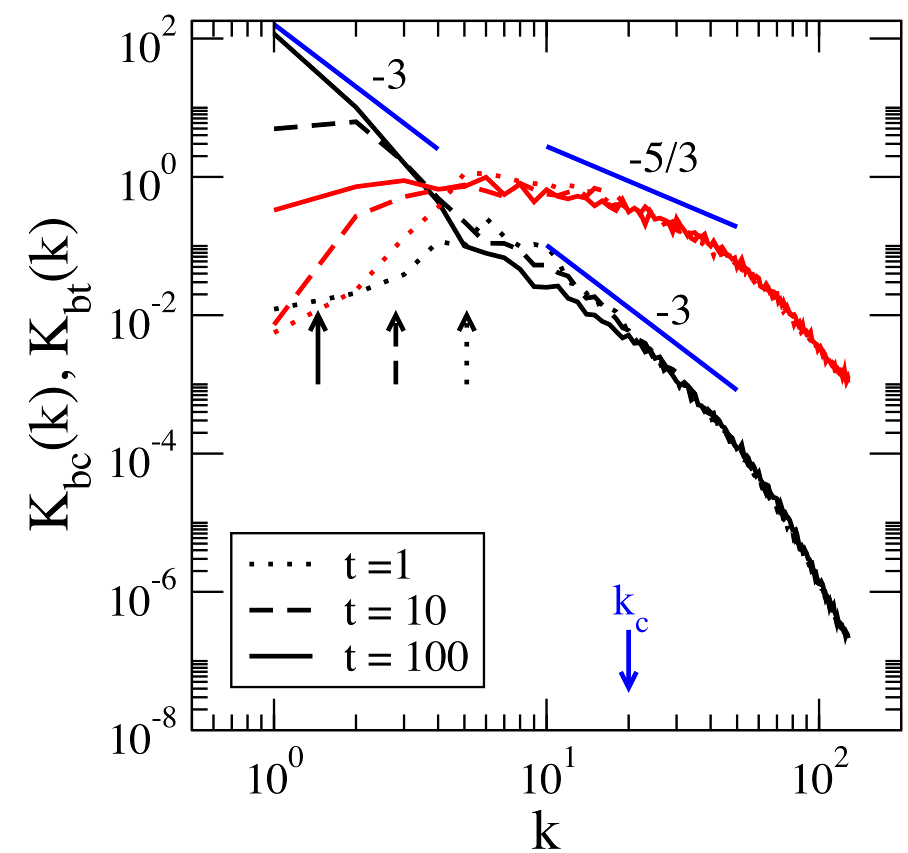

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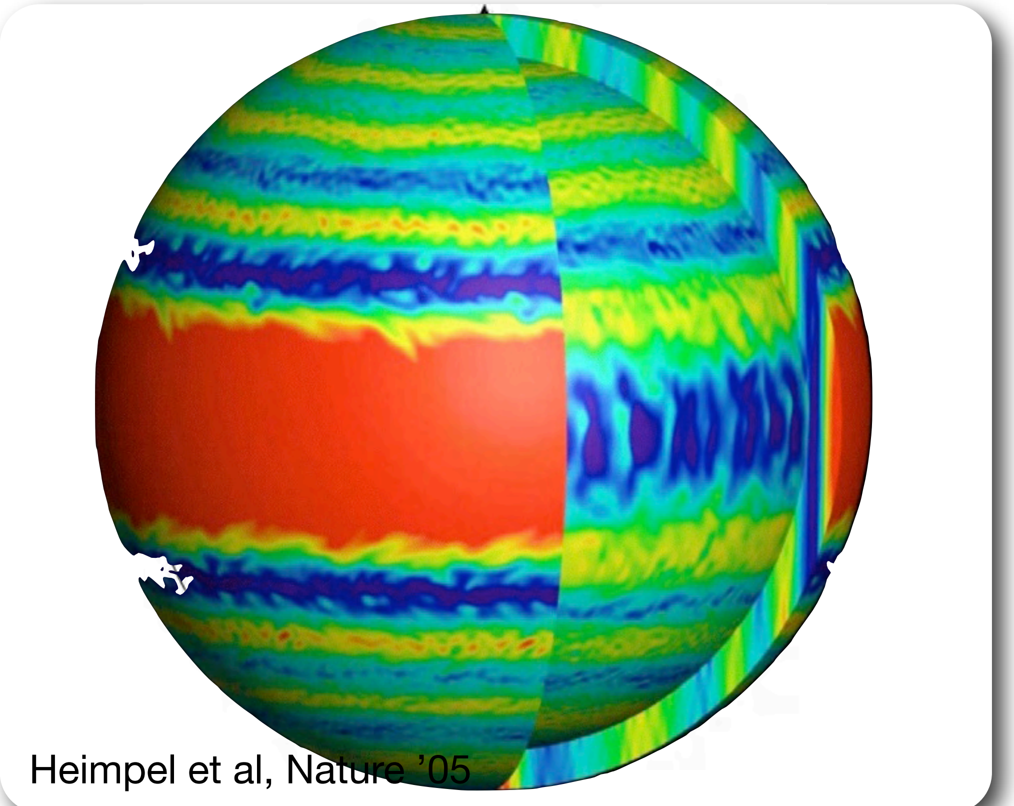
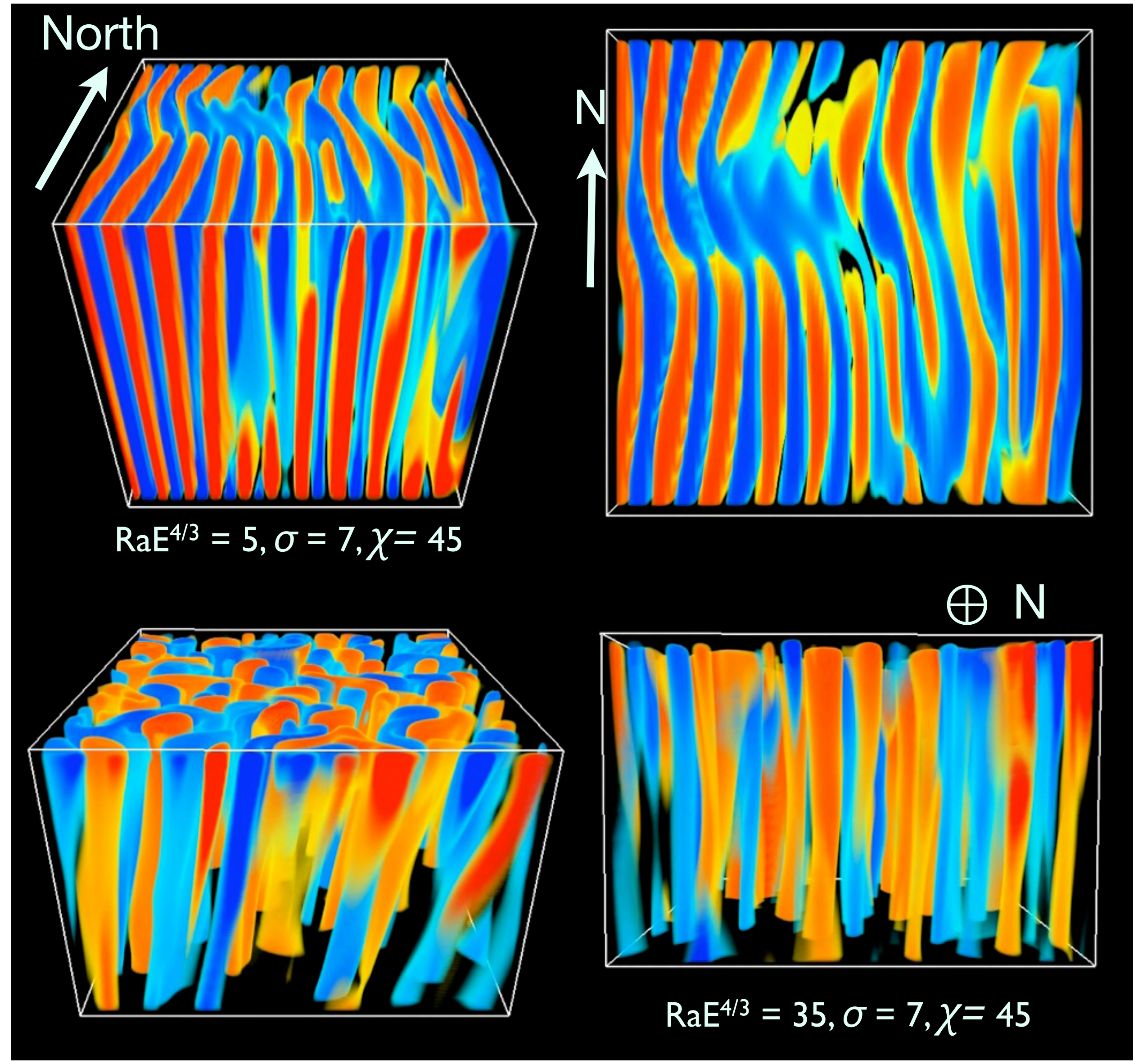


Heimpel et al, Nature '05



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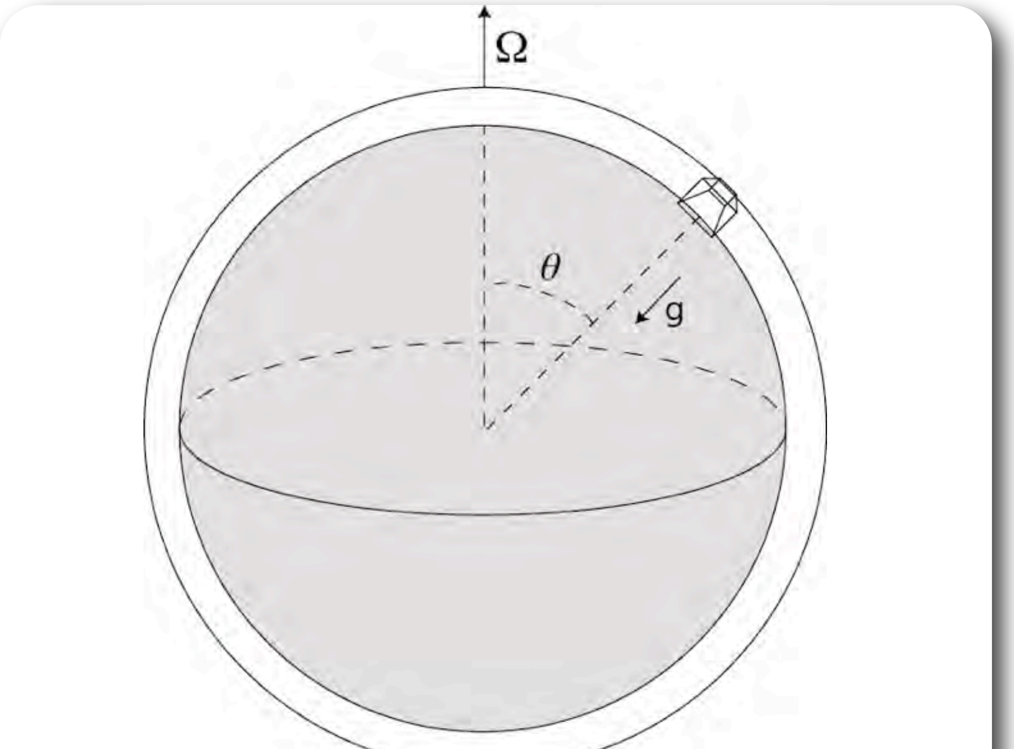
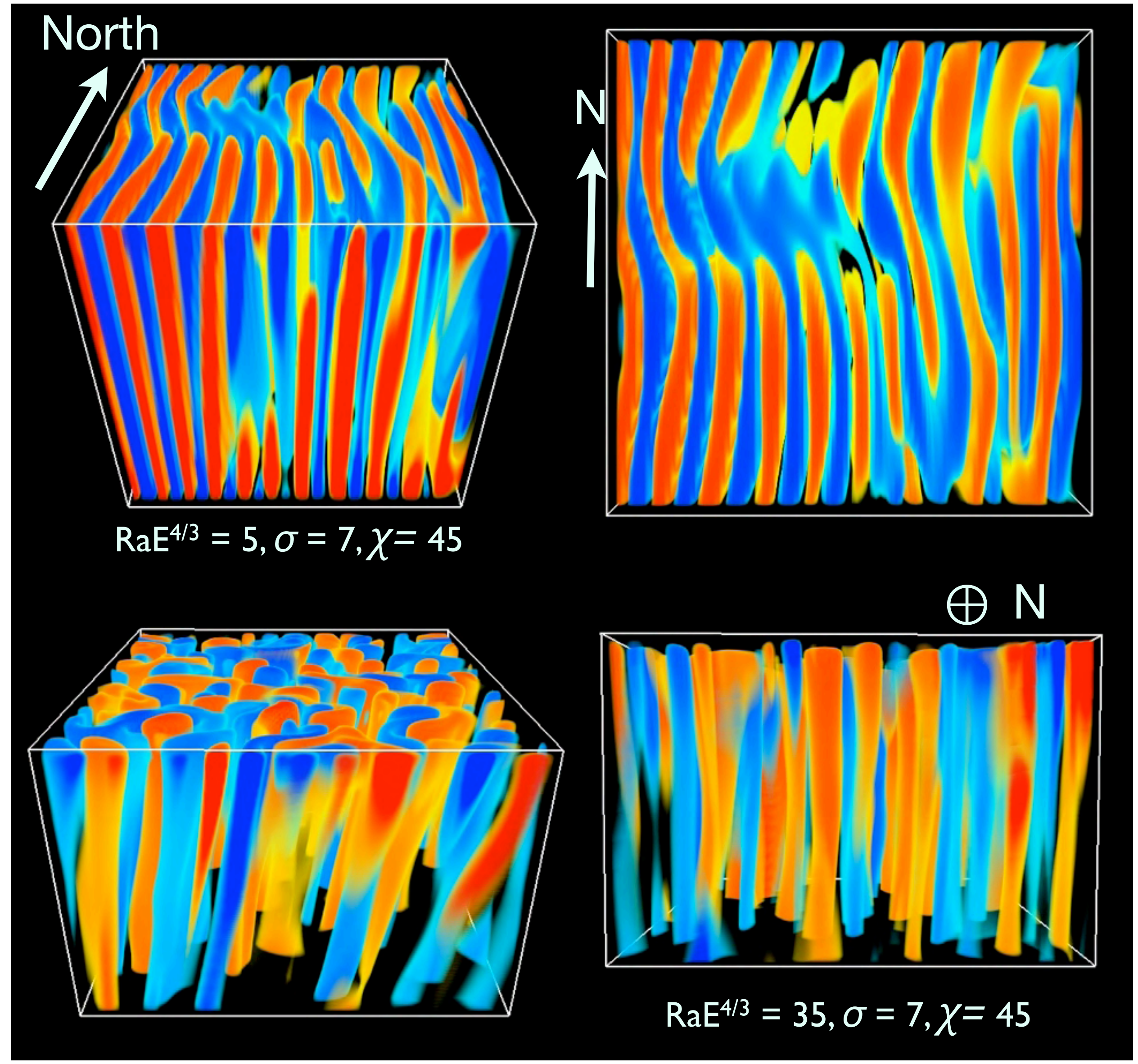


Calkins, J, Rubio '13



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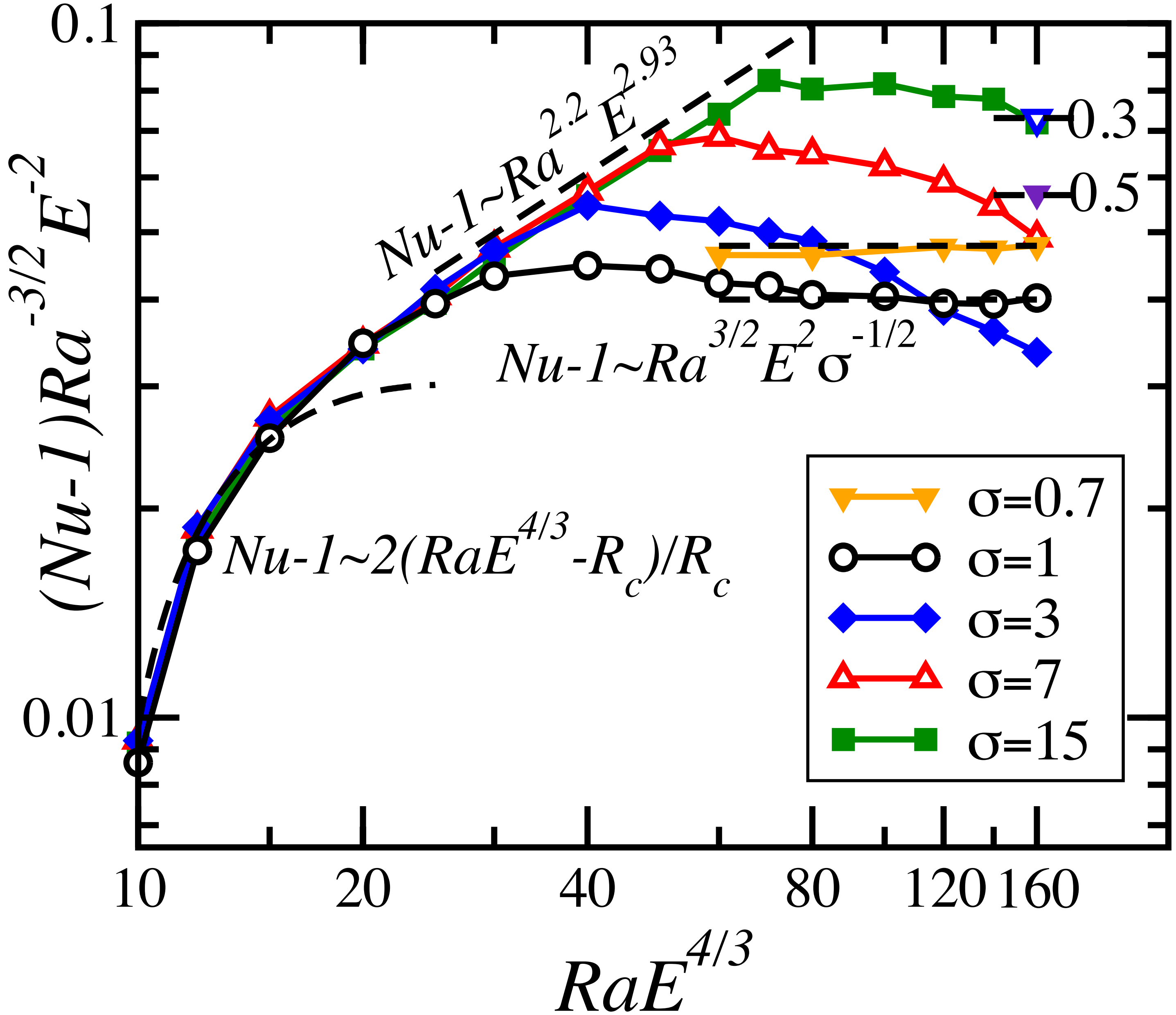
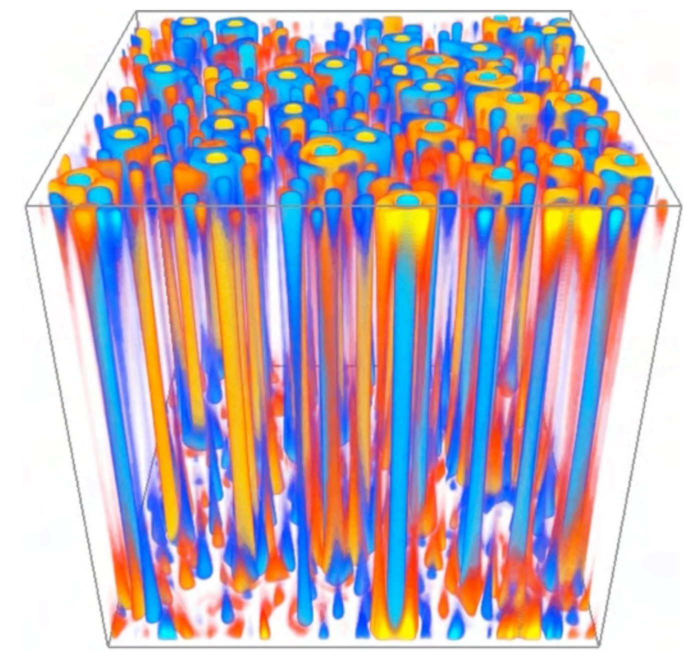
Low Ro Heat Transfer: $$Nu - 1 = \frac{1}{25}\sigma^{-\frac{1}{2}}\left(RaE^{\frac{4}{3}}\right)^{\frac{3}{2}}$$





Convective Taylor Columns

Nondimensional #'s:

$$Nu \equiv \frac{QH}{\rho_0 c_p \kappa \Delta T}, \quad Ra = \frac{g\alpha \Delta T H^3}{\nu\kappa}, \quad E = \frac{\nu}{fH^2}$$

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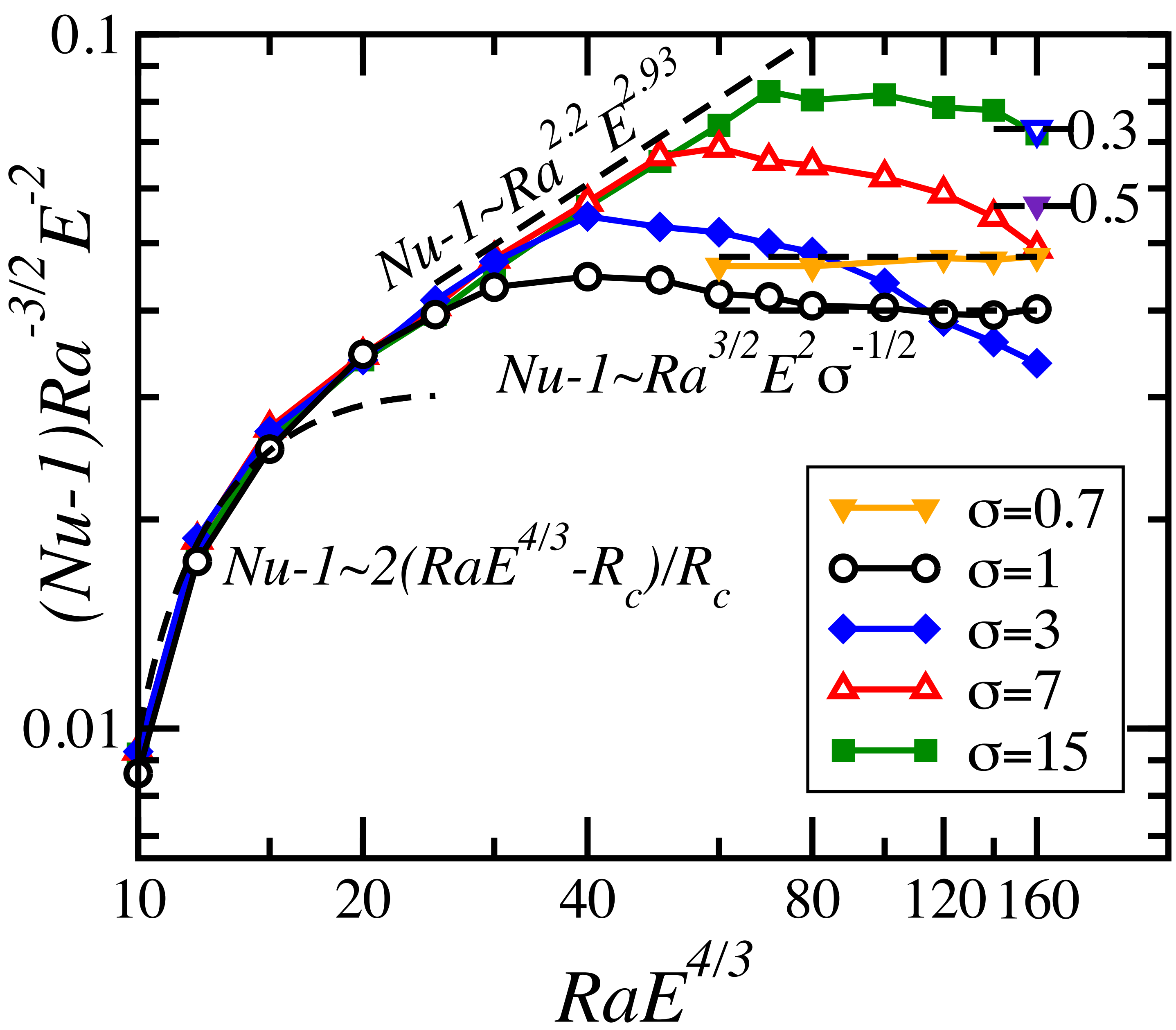

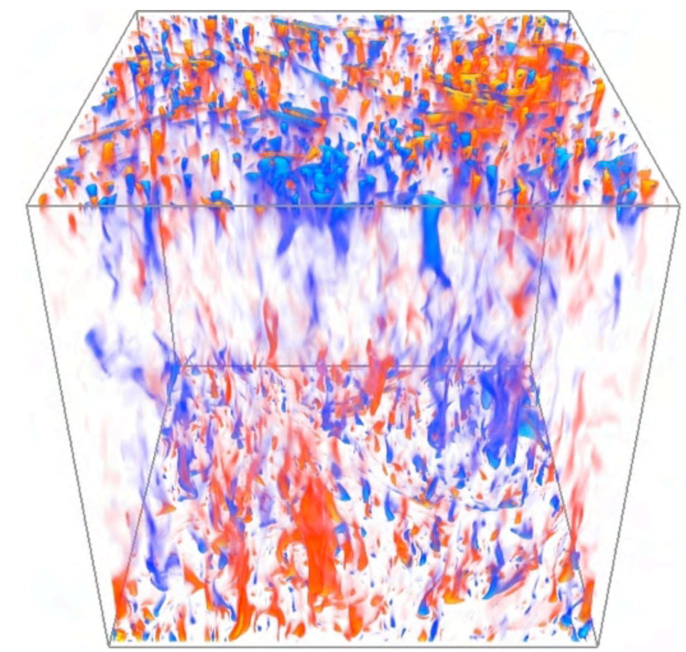

Geostrophic Turbulence

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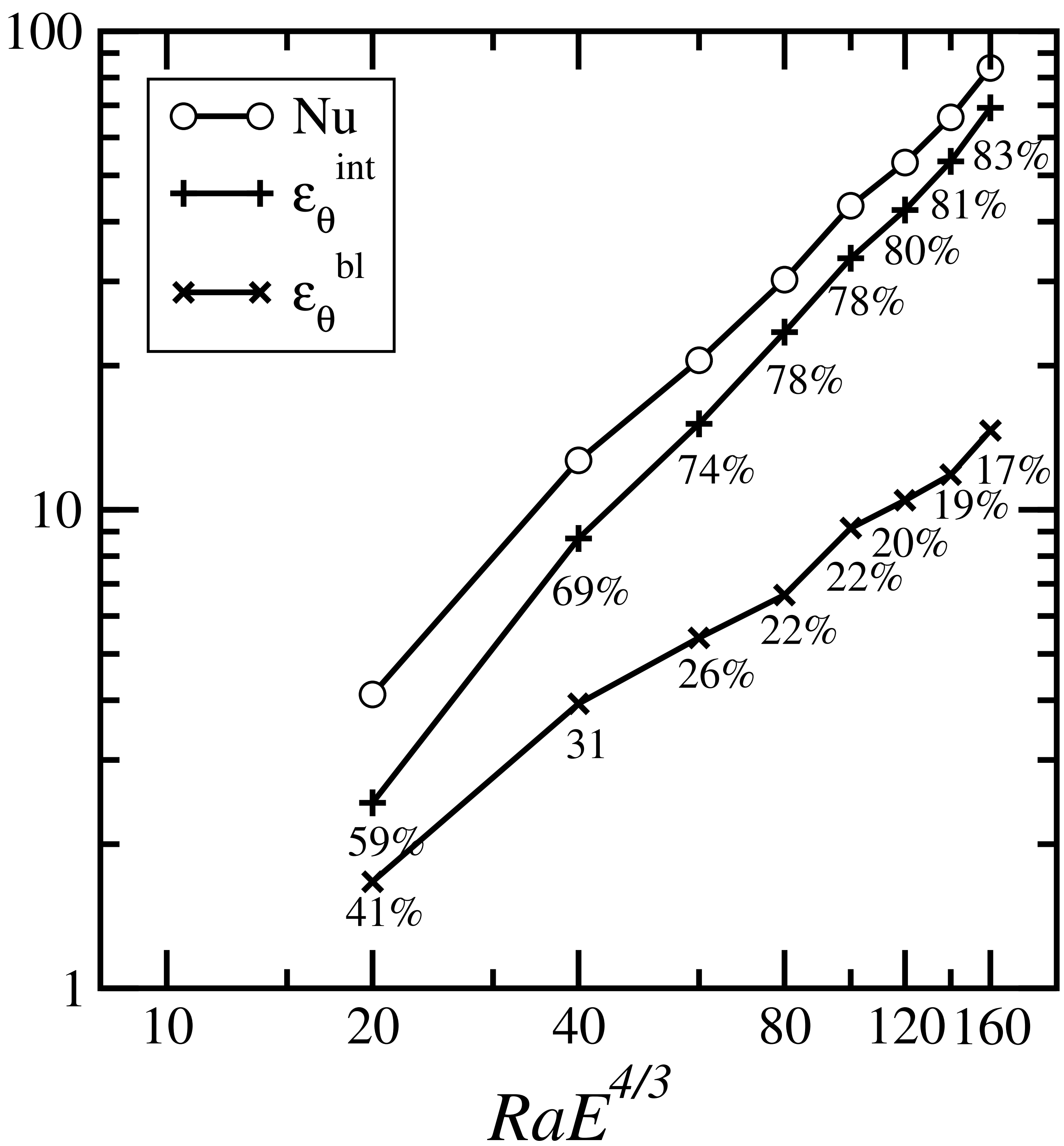

- turbulent interior controls heat transport (GL theory)

$$\mathcal{E}_\theta \approx \mathcal{E}_\theta^{int} = \left\langle \left|\partial_Z \overline{T}\right|^2 \right\rangle + \left\langle |\nabla_\perp \theta|^2 \right\rangle$$
$$\equiv Nu$$

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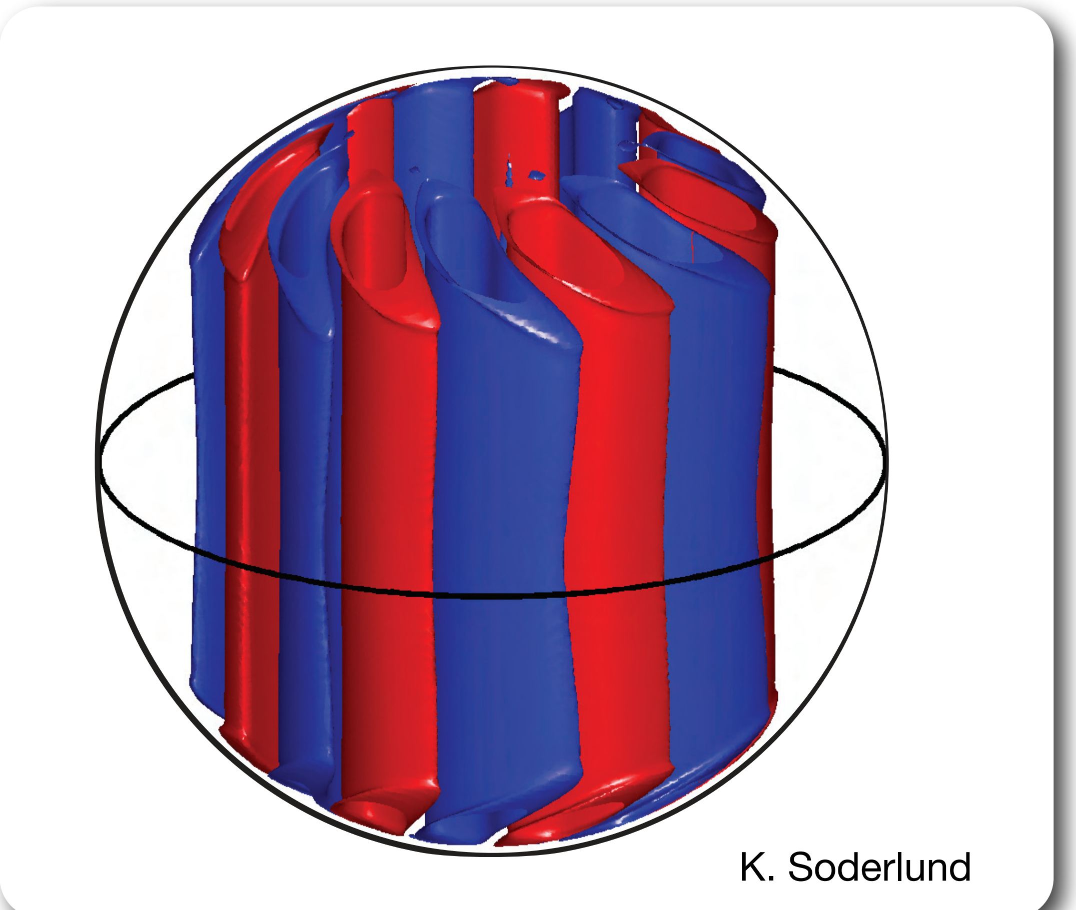
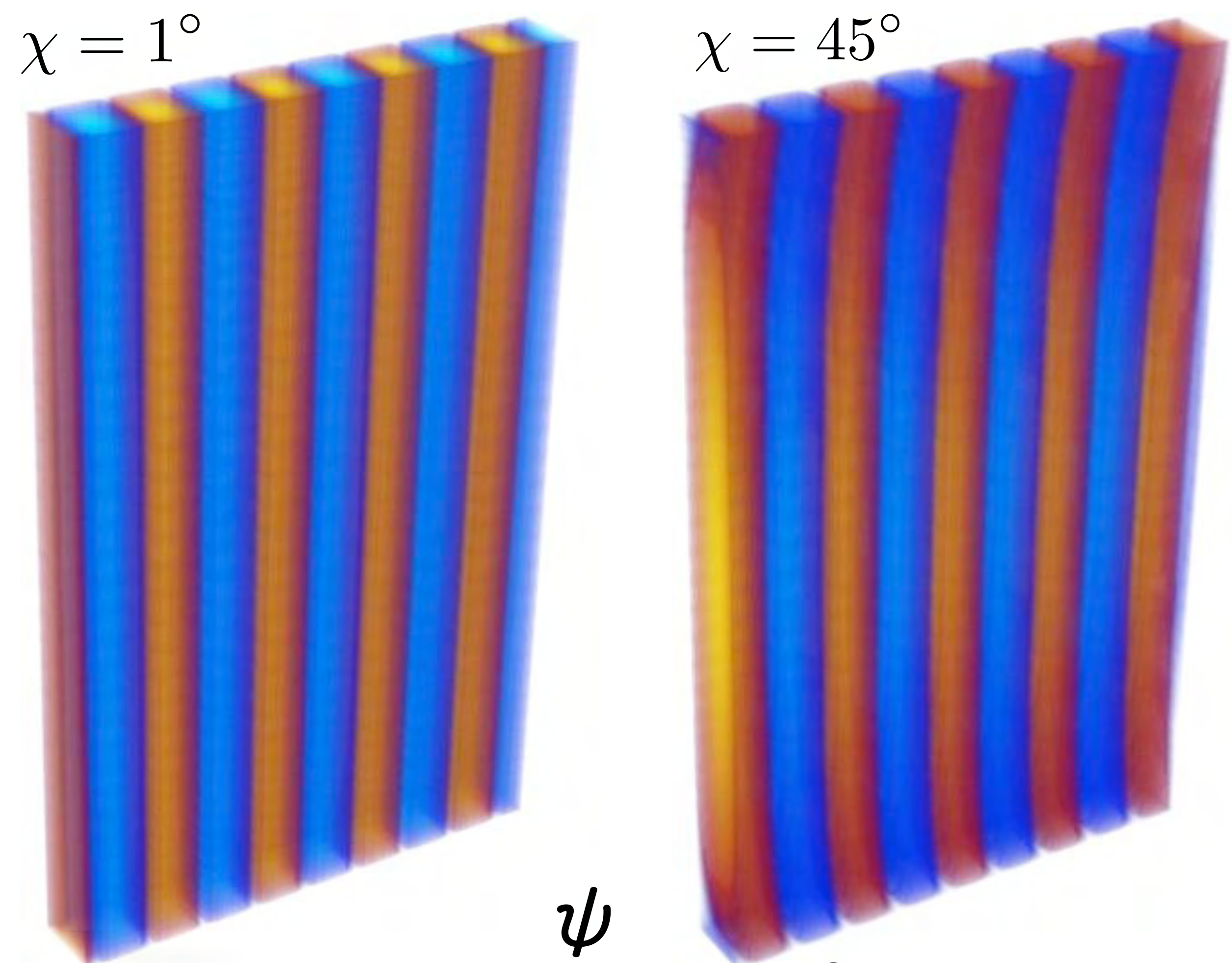


K. Soderlund



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  strong vertical motions, w~O(u)

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  Dynamics are fundamentally three dimensional

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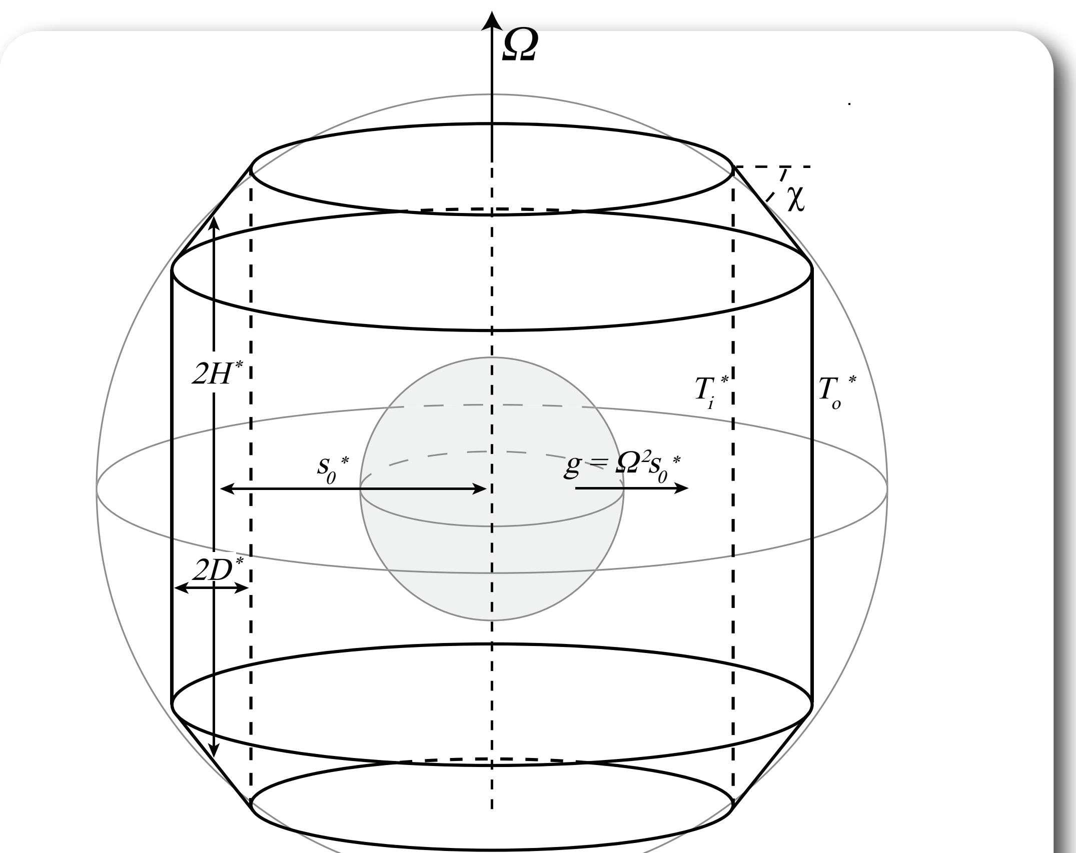
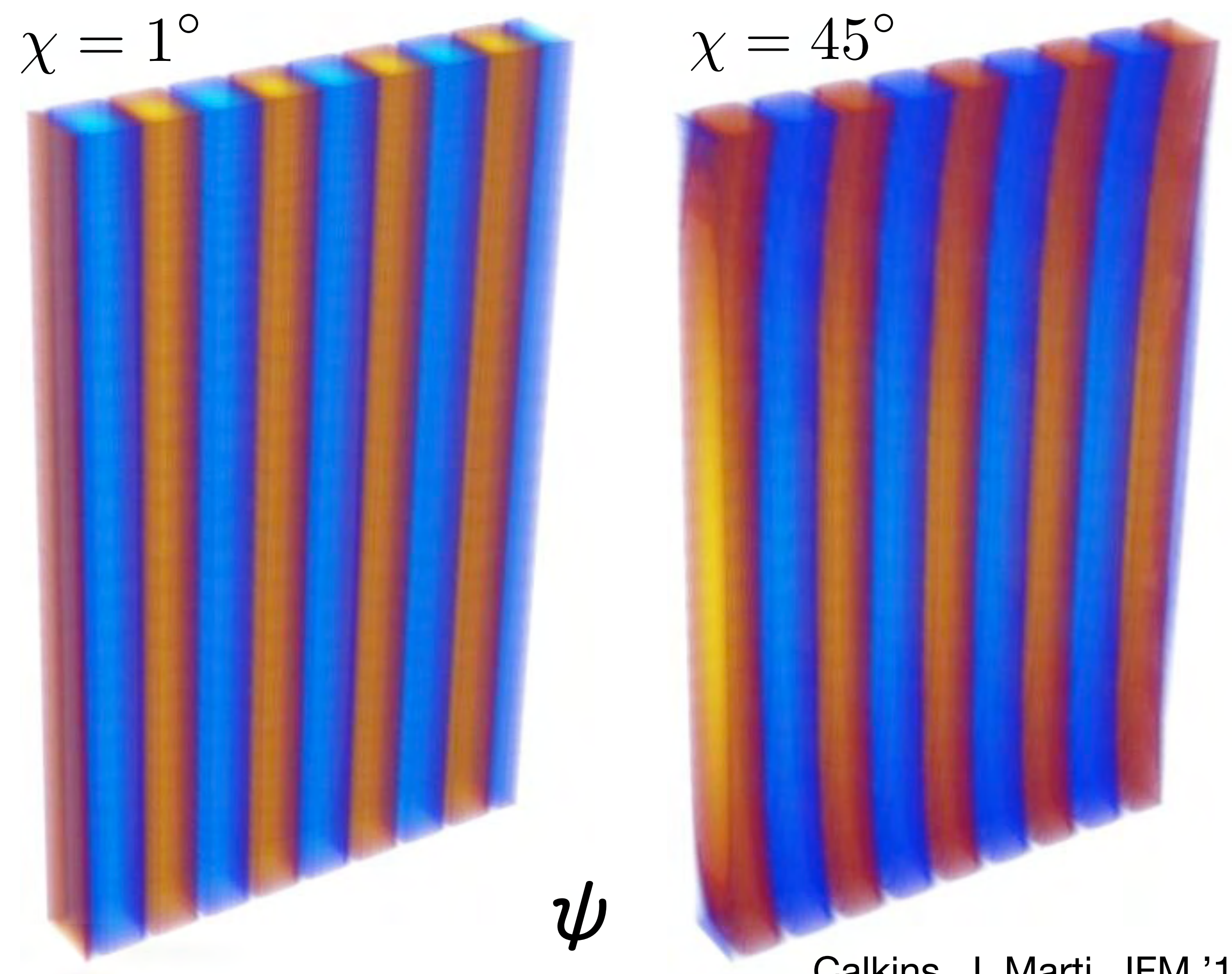


Busse JFM 1970



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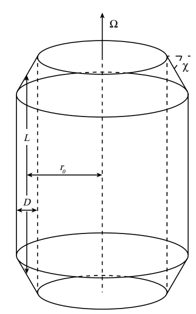

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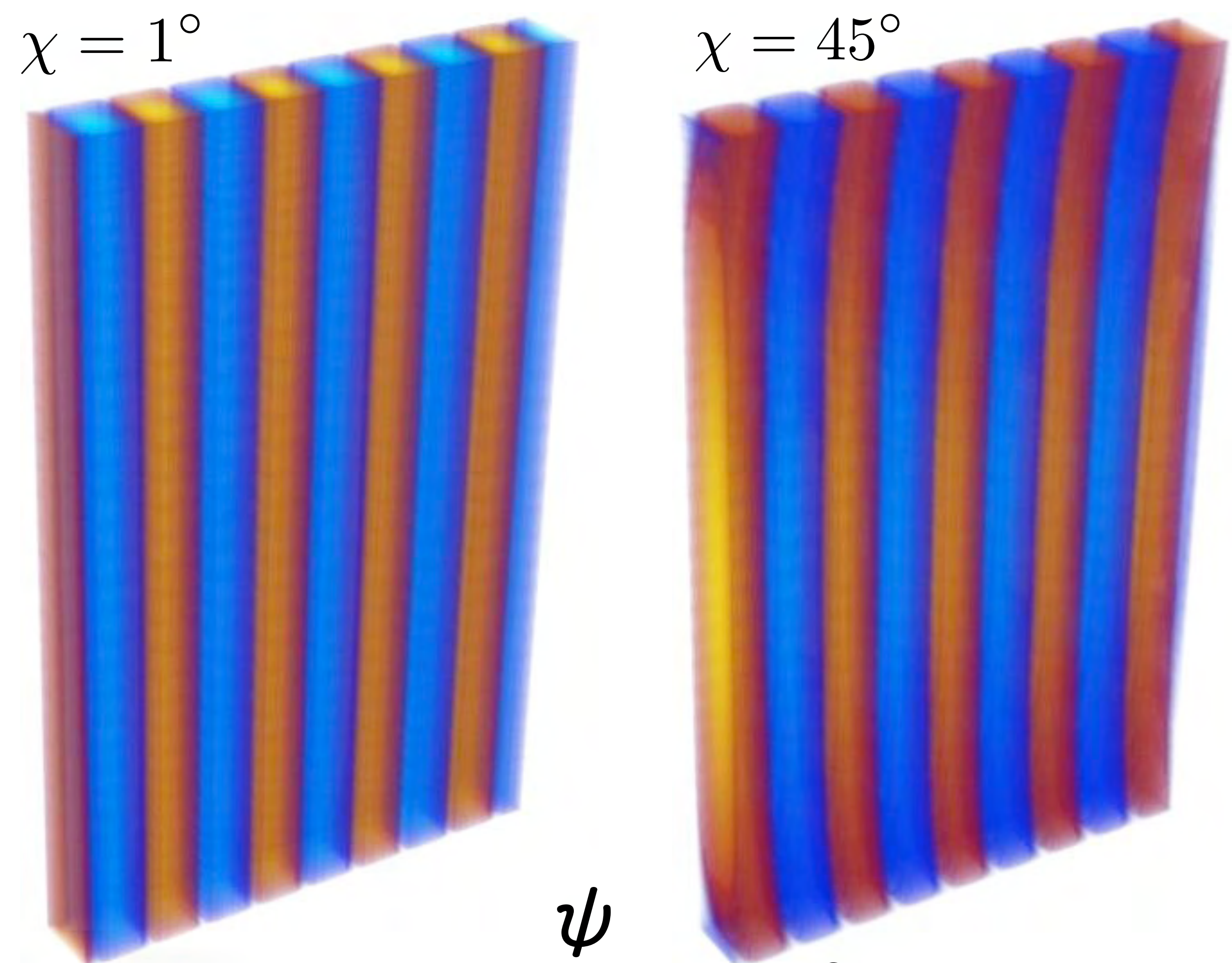

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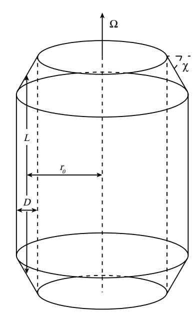

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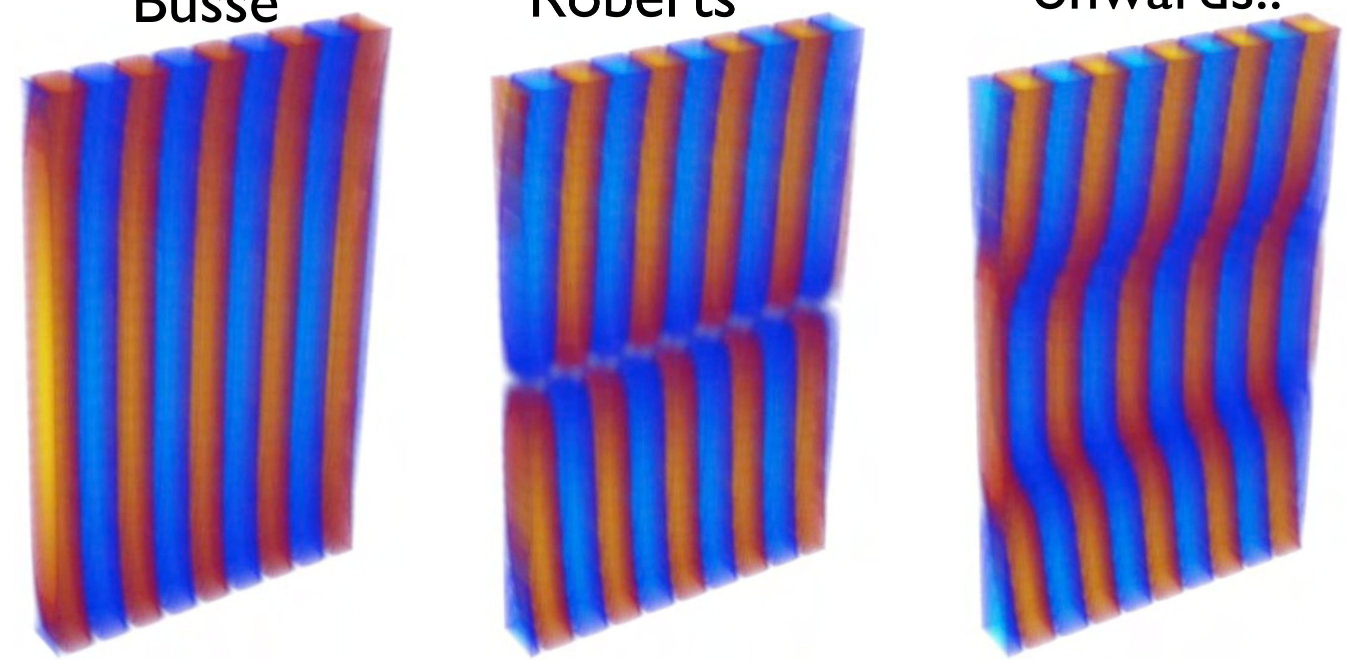
$\chi = 45^\circ$

Busse | Roberts | onwards..



Mode 1 | Mode 2 | Mode 3

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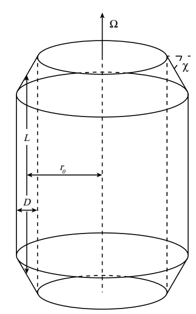

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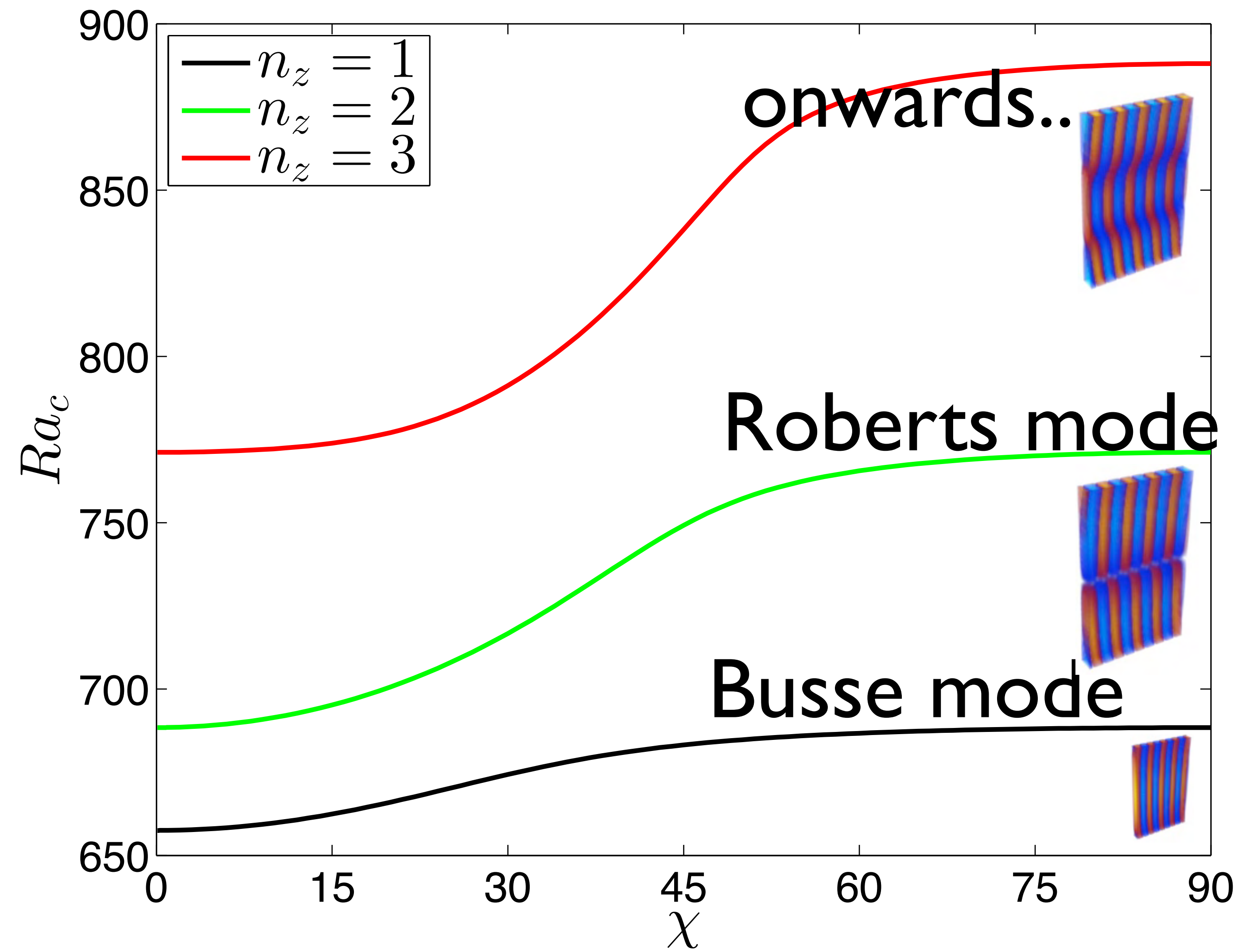

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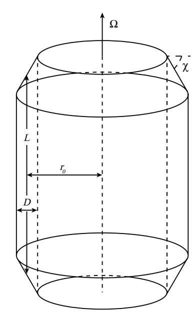

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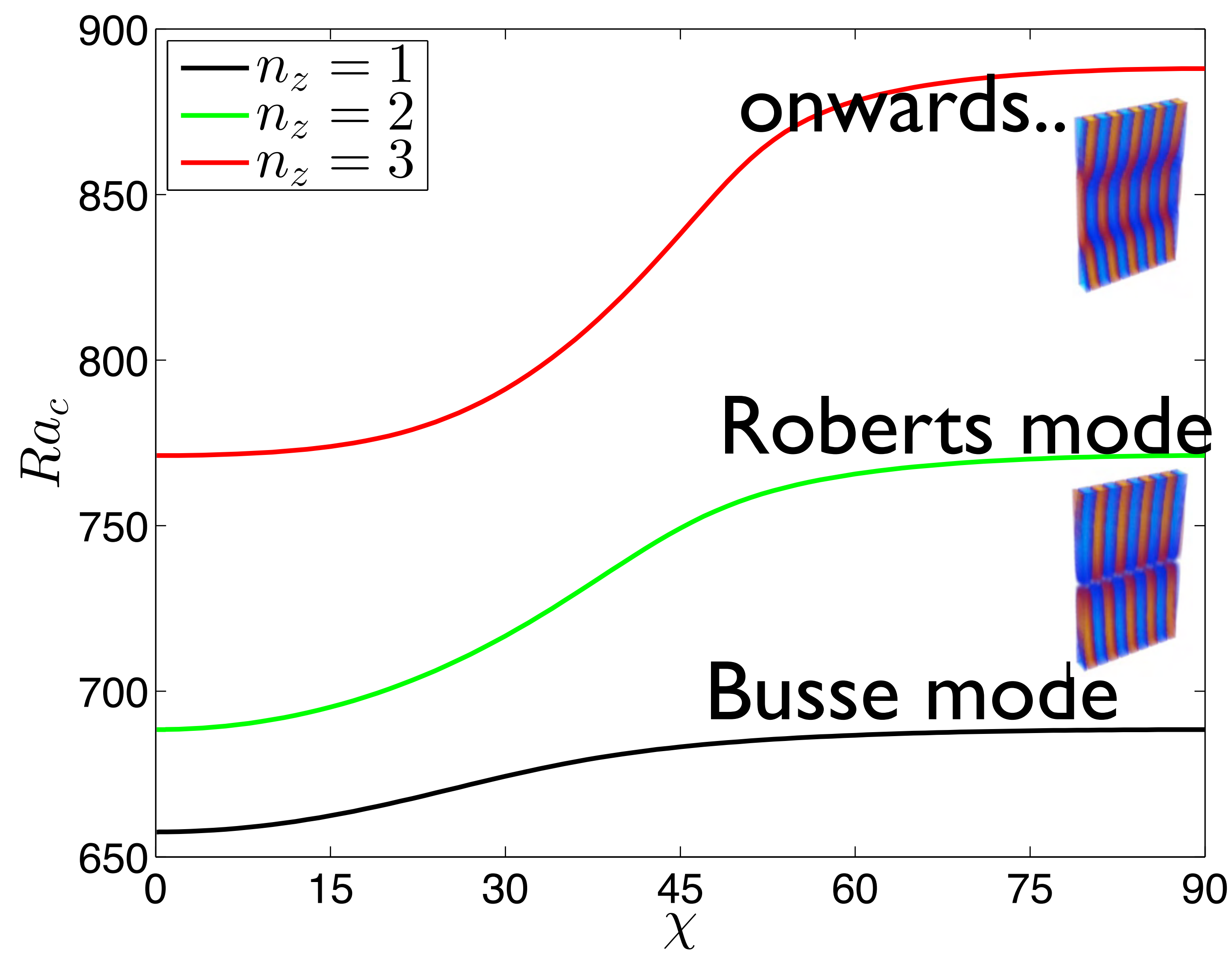

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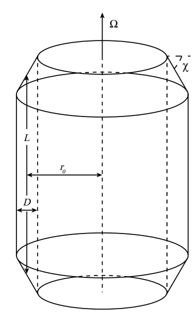

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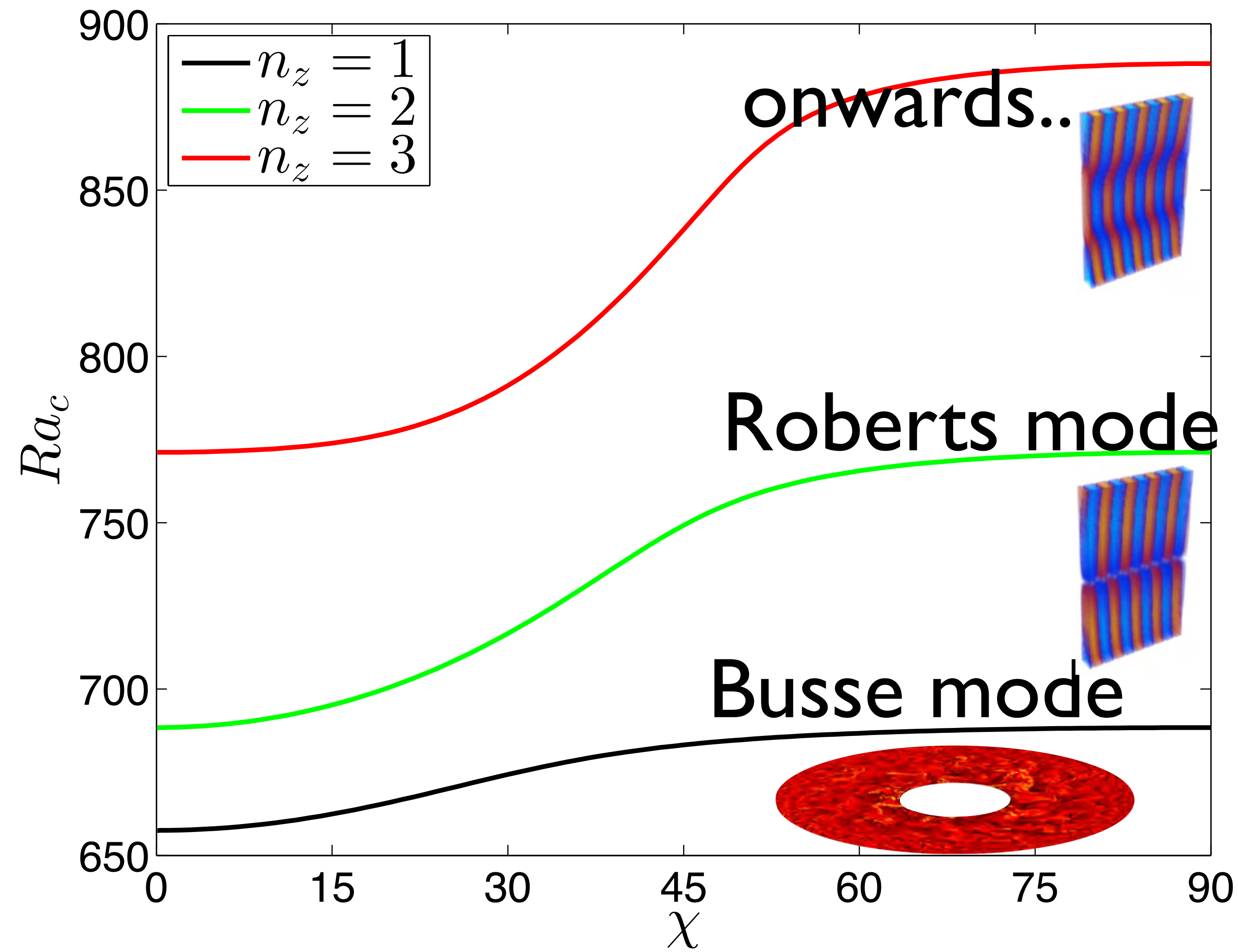

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$$\bar{\boldsymbol{U}} = \hat{\boldsymbol{Z}} \times \bar{\boldsymbol{\nabla}}_\perp \bar{P}, \quad \bar{\boldsymbol{\Theta}} = \partial_Z \bar{P}$$

$$\left(\frac{\partial}{\partial \bar{t}} + \bar{\boldsymbol{U}} \cdot \bar{\boldsymbol{\nabla}}\right) \partial_Z \bar{P} = 0 \qquad \frac{\partial \langle \bar{\boldsymbol{U}} \rangle}{\partial \bar{t}} + \bar{\boldsymbol{\nabla}} \cdot \langle \bar{\boldsymbol{U}} \otimes \bar{\boldsymbol{U}} + \boldsymbol{u} \otimes \boldsymbol{u} \rangle = -\bar{\boldsymbol{\nabla}} \langle \bar{\boldsymbol{\Pi}} \rangle$$

Baroclinic Dynamics — Barotropic Dynamics

Non-Hydrostatic Dynamics

$$\hat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -\boldsymbol{\nabla}_\perp p, \quad p = -\psi$$

$$\left(\partial_t + \bar{\boldsymbol{U}} \cdot \boldsymbol{\nabla}_\perp\right) \nabla_\perp^2 \psi + J(\psi, \nabla_\perp^2 \psi) + \partial_Z w = \frac{1}{\mathrm{Re}} \nabla_\perp^4 \psi$$

$$\left(\partial_t + \bar{\boldsymbol{U}} \cdot \boldsymbol{\nabla}_\perp\right) w + J(\psi, w) - \partial_Z \psi = \theta + \frac{1}{\mathrm{Re}} \nabla_\perp^2 w$$

$$\left(\partial_t + \bar{\boldsymbol{U}} \cdot \boldsymbol{\nabla}_\perp\right) \theta + J(\psi, \theta) + \boldsymbol{\nabla}_\perp \psi \cdot \partial_Z \bar{\boldsymbol{U}} + w \partial_Z \bar{\Theta} = \frac{1}{\mathrm{Pe}} \nabla_\perp^2 \theta$$

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{\mathrm{Pe}} \partial_Z^2\right) \bar{T} = -\partial_Z \bar{F}$$

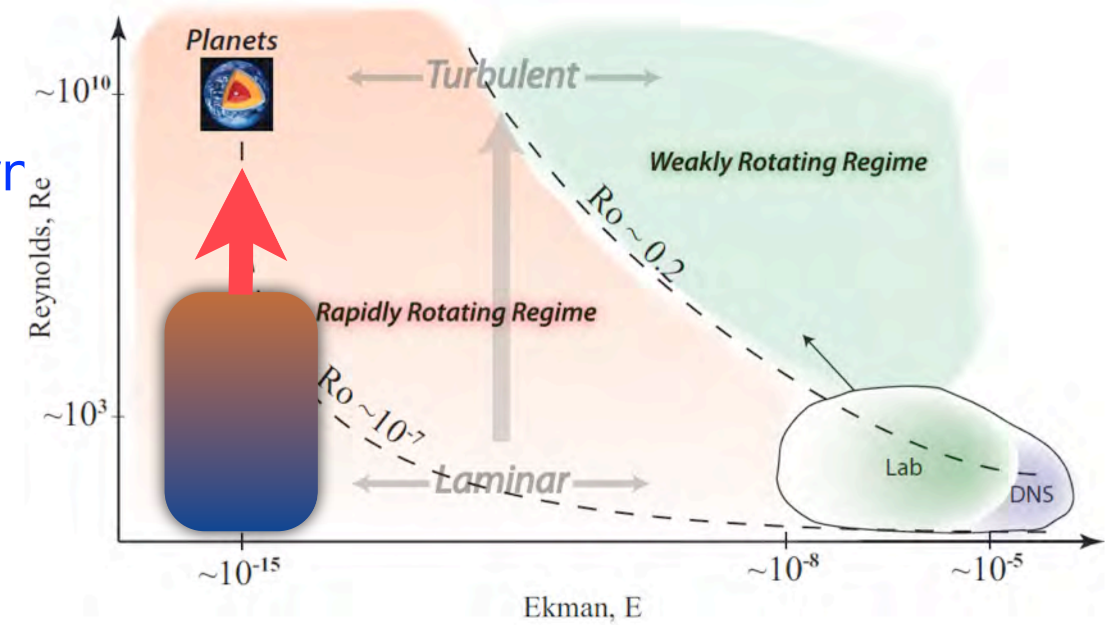
Global Mean Temperature & Flux

$$\bar{T} = \lim_{\bar{t} \to \infty} \frac{1}{\bar{t}} \int_0^{\bar{t}} \frac{1}{|A|} \int_A \bar{\Theta} \, \mathrm{d}X \mathrm{d}Y \mathrm{d}\bar{t}' \qquad \bar{F} = \lim_{\bar{t} \to \infty} \frac{1}{\bar{t}} \int_0^{\bar{t}} \frac{1}{|A|} \left[\int_A \overline{w\theta} \, \mathrm{d}X \mathrm{d}Y - \oint_{\partial A} \bar{\boldsymbol{U}} \cdot \mathrm{d}\boldsymbol{l}\right] \mathrm{d}\bar{t}'$$

Outlook for 3D QG

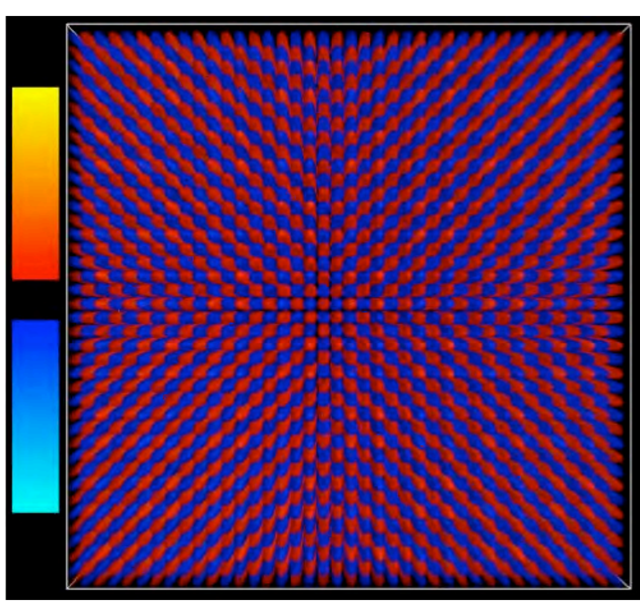
Thank you

- Reduced PDE’s well suited to QG dynamics, computationally less challenging.
- Incompressible aDNS (“a”symptotic)
  - Investigate route to turbulence: columnar breakdown
  - Mean flow generation: inverse turbulent cascade?
  - Efficiency of heat transport: scaling laws

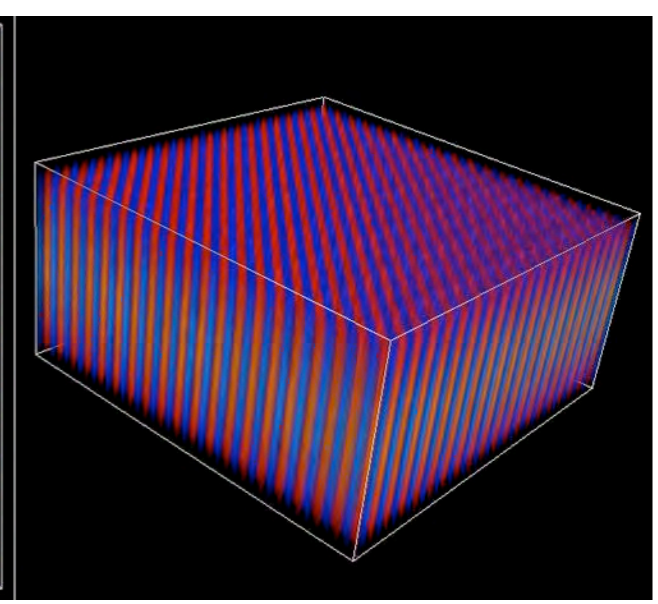


- Anelastic (stratification) aDNS Simulations
- Coupling to reduced planetary scale dynamics, required by MHD

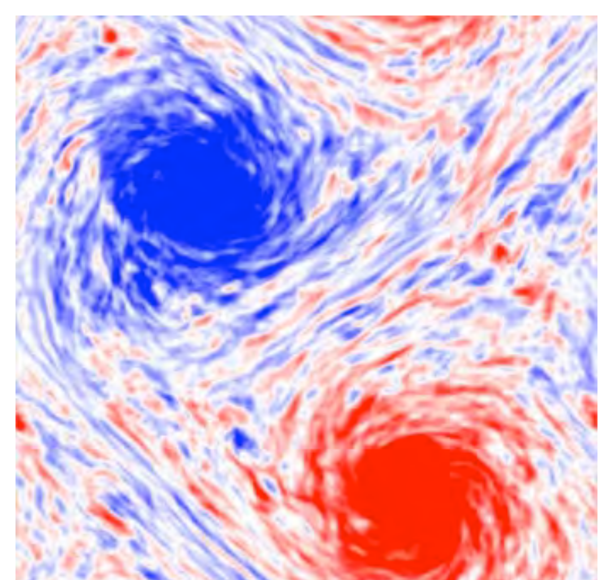
Sprague et al JFM ’06



Groom et al PRL ‘10



Julien et al GAFD ‘12



Julien et al PRL ‘12