# Sub-grid scale model with structure effects incorporated through the helicity


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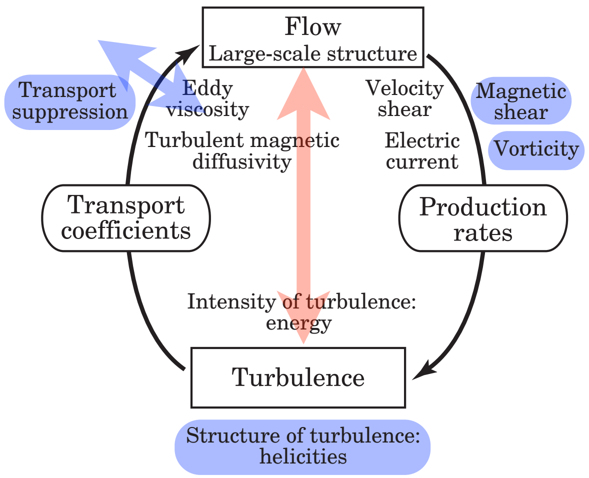
In collaboration with Akira YOSHIZAWA


turbulent electromotive force

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -\beta \mathbf{J} + \gamma \boldsymbol{\Omega} + \alpha \mathbf{B}$$

Reynolds (+ turb. Maxwell) stress

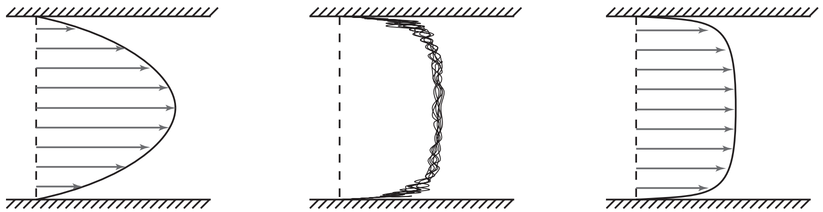
$$\left[ \langle u'^\alpha u'^\beta - b'^\alpha b'^\beta \rangle \right]_{\mathrm{D}}$$
$$= -\nu_{\mathrm{K}} \mathcal{S}^{\alpha\beta} + \nu_{\mathrm{M}} \mathcal{M}^{\alpha\beta} + \left[ \Gamma^\alpha \Omega^\beta + \Gamma^\beta \Omega^\alpha \right]_{\mathrm{D}}$$

$\mathcal{S}$ : mean velocity strain $\mathcal{M}$ : mean magnetic strain

# Swirling flow and turbulence

(Yokoi & Yoshizawa, PoF, 1993)

laminar | turbulent | turbulent mean



Mean velocity $\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\mathbf{U} = -\nabla P - \nabla\cdot\boldsymbol{\mathcal{R}} + \nu\nabla^2\mathbf{U}$

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &\equiv \left\langle u'^{\alpha} u'^{\beta}\right\rangle \\ &= \frac{2}{3}K\delta^{\alpha\beta} - \nu_{\mathrm{T}}\mathcal{S}^{\alpha\beta}\end{aligned}$$

$\nu_{\mathrm{K}} = C_\nu \tau K$ $\qquad$ $\tau$ : Time scale of turbulence

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)U^\alpha = -\frac{\partial P}{\partial x^\alpha} + \frac{\partial}{\partial x^a}\left[(\nu + \nu_{\mathrm{T}})\left(\frac{\partial U^\alpha}{\partial x^a} + \frac{\partial U^a}{\partial x^\alpha}\right)\right]$$

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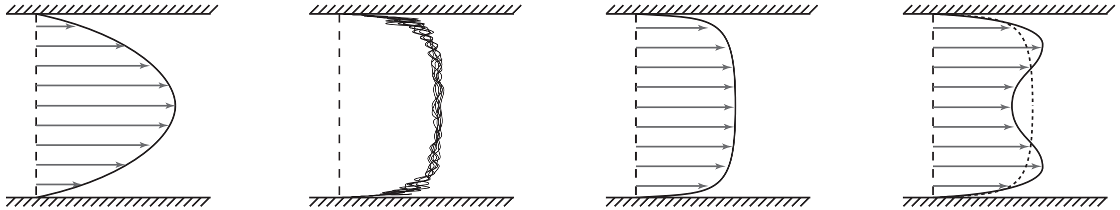

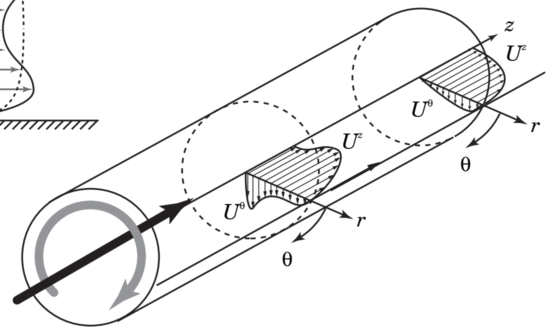


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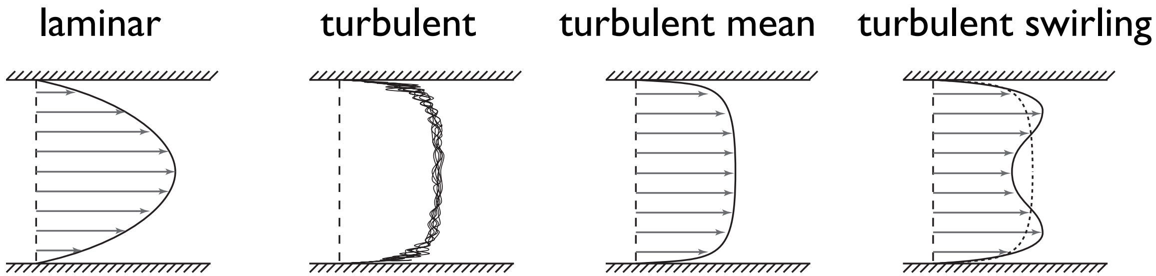


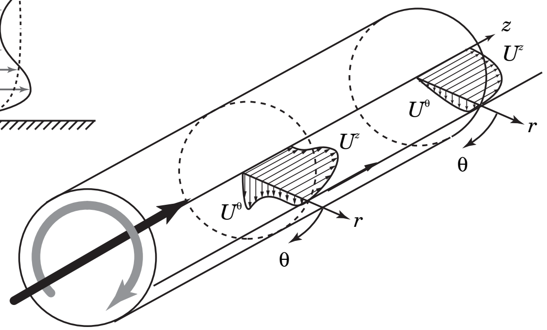


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$\nu_{\mathrm{K}} = C_\nu \tau K$ $\quad\eta = C_\eta \tau K^3/\epsilon^2$ $\quad\tau$ : Time scale of turbulence

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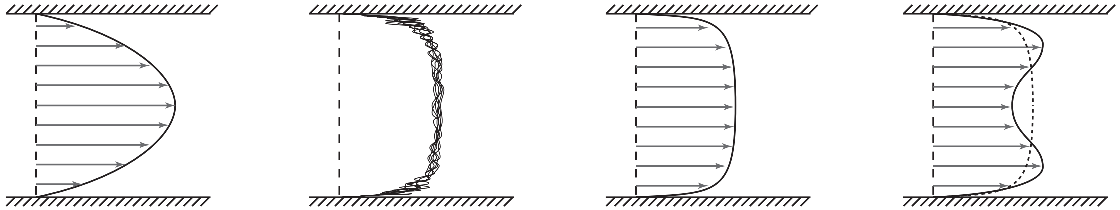

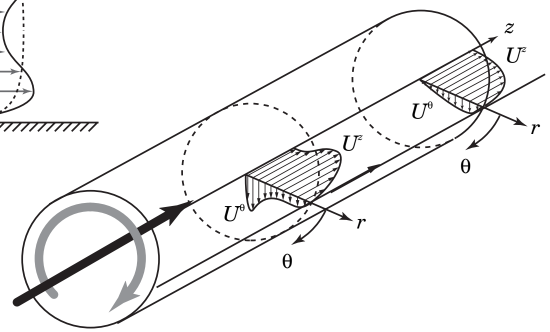


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$\nu_{\mathrm{K}} = C_\nu \tau K \qquad \eta = C_\eta \tau K^3 / \epsilon^2 \qquad \tau$ : Time scale of turbulence

$$\begin{aligned}\left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) U^{\alpha} = &- \frac{\partial P}{\partial x^{\alpha}} + \frac{\partial}{\partial x^{a}} \left[ (\nu + \nu_{\mathrm{T}}) \left( \frac{\partial U^{\alpha}}{\partial x^{a}} + \frac{\partial U^{a}}{\partial x^{\alpha}} \right) \right] \\ &+ \frac{\partial}{\partial x^{a}} \left\{ \eta \left[ \Omega^{\alpha} \frac{\partial H}{\partial x^{a}} + \Omega^{a} \frac{\partial H}{\partial x^{\alpha}} - \frac{2}{3} \delta^{\alpha a} \left( \boldsymbol{\Omega} \cdot \nabla \right) H \right] \right\}\end{aligned}$$

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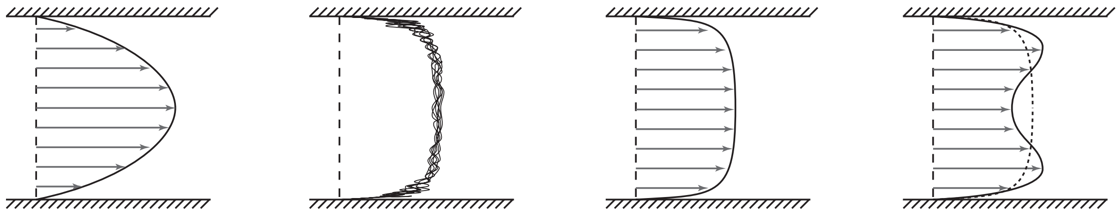

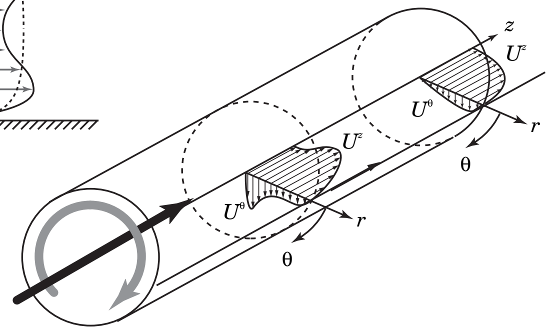


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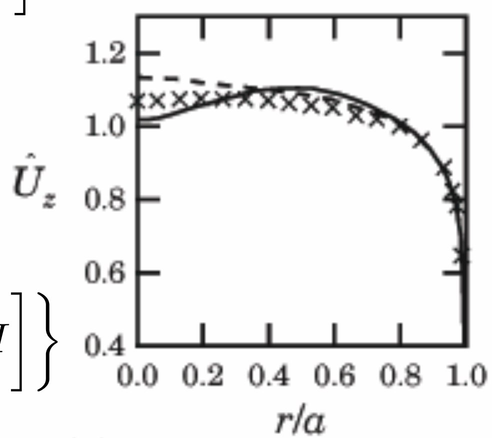
$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)U^{\alpha} &= -\frac{\partial P}{\partial x^{\alpha}} + \frac{\partial}{\partial x^{a}}\left[\left(\nu+\nu_{\mathrm{T}}\right)\left(\frac{\partial U^{\alpha}}{\partial x^{a}} + \frac{\partial U^{a}}{\partial x^{\alpha}}\right)\right] \\ &\quad + \frac{\partial}{\partial x^{a}}\left\{\eta\left[\Omega^{\alpha}\frac{\partial H}{\partial x^{a}} + \Omega^{a}\frac{\partial H}{\partial x^{\alpha}} - \frac{2}{3}\delta^{\alpha a}\left(\boldsymbol{\Omega}\cdot\nabla\right)H\right]\right\}\end{aligned}$$



# Structure effect

## Classical Smagorinsky model

### Problem of constant adjustment

SGS viscosity

$$\nu_{\mathrm{S}} = (C_{\mathrm{S}}\Delta)^2 S$$

$C_{\mathrm{S}}$ : Smagorinsky constant

| | Smagorinsky constant $C_{\mathrm{S}}$ |
|---|---|
| Isotropic flow | $0.2$ |
| Mixing-layer flow | $0.15$ |
| Channel flow | $0.1$ |

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| | Smagorinsky constant $C_S$ | Dissipative nature |
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| Isotropic flow | $0.2$ | more |
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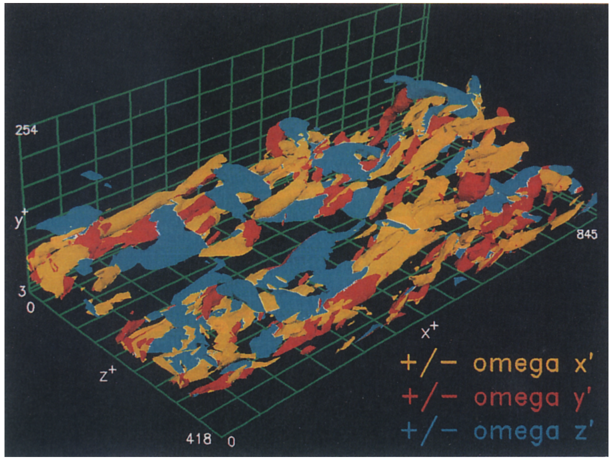


Fluctuating vorticity
(Robinson, Kline & Spalart 1988)

| | Smagorinsky constant $C_S$ | Dissipative nature | Vortical structures |
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| Isotropic flow | $0.2$ | more | weak |
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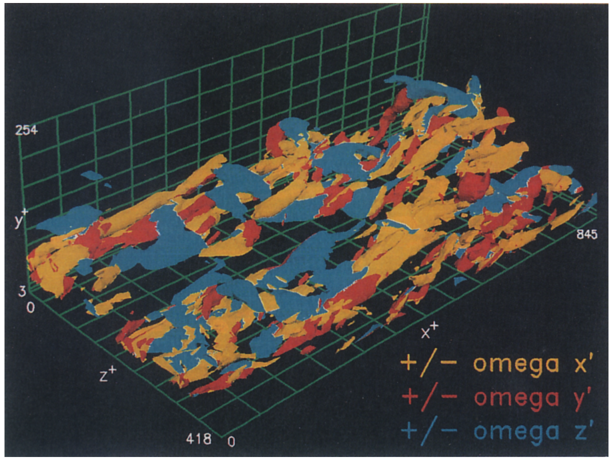


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Coherent vortical structures
may be related to the less dissipative nature

# Helicity SGS model

## Grid-Scale (GS) velocity equation

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \overline{\mathbf{u}} = -\nabla \overline{p} - \nabla \cdot \boldsymbol{\mathcal{R}} + \nu \nabla^2 \overline{\mathbf{u}}$$

SGS stress

$$\mathcal{R}^{\alpha\beta} \equiv \overline{u^\alpha u^\beta} - \overline{u}^\alpha \overline{u}^\beta$$

$$= \frac{2}{3} K_\mathrm{S} \delta^{\alpha\beta} - \nu_\mathrm{S} \mathcal{S}^{\alpha\beta} + \eta_\mathrm{S} \left[\overline{\omega}^\alpha \frac{\partial H_\mathrm{S}}{\partial x^\beta} + \overline{\omega}^\beta \frac{\partial H_\mathrm{S}}{\partial x^\alpha} - \frac{2}{3} \delta^{\alpha\beta} \left(\overline{\boldsymbol{\omega}} \cdot \nabla\right) H_\mathrm{S}\right]$$

SGS viscosity $\nu_\mathrm{S} = C_\mathrm{S} \Delta K_\mathrm{S}^{1/2}$ Helicity-related coefficient $\eta_\mathrm{S} = C_{\eta\mathrm{S}} \Delta^3 K_\mathrm{S}^{-1/2}$

# Two-equation model

SGS *K* equation $$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) K_\mathrm{S} = -\mathcal{R}^{ab} \frac{\partial \overline{u}^b}{\partial x^a} - \epsilon_\mathrm{S} + \nabla \cdot \mathbf{T}_\mathrm{S}$$

$$K_\mathrm{S} = \frac{1}{2} \overline{\mathbf{u}'^2} \qquad \epsilon_\mathrm{S} = C_{\epsilon\mathrm{S}} \frac{K_\mathrm{S}^{3/2}}{\Delta} \qquad \mathbf{T}_\mathrm{S} = \frac{\nu}{\sigma_\mathrm{S}} \nabla K_\mathrm{S}$$

SGS *H*S equation $$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) H_\mathrm{S} = -\mathcal{R}^{ab} \frac{\partial \overline{\omega}^b}{\partial x^a} + \overline{\omega}^a \frac{\partial \mathcal{R}^{ab}}{\partial x^b} - \epsilon_\mathrm{HS} + \nabla \cdot \left(K_\mathrm{S} \overline{\boldsymbol{\omega}} + \mathbf{T}_\mathrm{HS}\right)$$

$$H_\mathrm{S} = \overline{\mathbf{u}' \cdot \boldsymbol{\omega}'} \qquad \epsilon_\mathrm{HS} = C_{\epsilon\mathrm{H}} \frac{H_\mathrm{S}}{K_\mathrm{S}/\epsilon_\mathrm{S}} = C_{\epsilon\mathrm{H}} \frac{\epsilon_\mathrm{S} H_\mathrm{S}}{K_\mathrm{S}} \qquad \mathbf{T}_\mathrm{HS} = \frac{\nu_\mathrm{S}}{\sigma_\mathrm{HS}} \nabla H_\mathrm{S}$$

# One-equation model

$$\{\nu_\mathrm{S},\ \eta_\mathrm{S}\} = \{\nu_\mathrm{S},\ \eta_\mathrm{S}\}(\Delta, S) \qquad \{K_\mathrm{S},\ \epsilon_\mathrm{S}\} = \{K_\mathrm{S},\ \epsilon_\mathrm{S}\}(\Delta, S)$$

SGS turbulent energy $K_\mathrm{S}$ $\quad K_\mathrm{S} = \Delta^2 S^2$

SGS viscosity $\quad \nu_\mathrm{S} = C_\mathrm{S}^2 \Delta K_\mathrm{S}^{1/2} = (C_\mathrm{S}\Delta)^2 S$

Helicity-related coefficient $\quad \eta_\mathrm{S} = C_{\eta\mathrm{S}} \Delta^3 K_\mathrm{S}^{-1/2} = C_{\eta\mathrm{S}} \dfrac{\Delta^2}{S}$

SGS Reynolds stress

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &= \frac{2}{3} K_\mathrm{S} \delta^{\alpha\beta} - \nu_\mathrm{S} \mathcal{S}^{\alpha\beta} + \eta_\mathrm{S} \left[ \overline{\omega}^\alpha \frac{\partial H_\mathrm{S}}{\partial x^\beta} + \overline{\omega}^\beta \frac{\partial H_\mathrm{S}}{\partial x^\alpha} - \frac{2}{3} \delta^{\alpha\beta} \left( \overline{\boldsymbol{\omega}} \cdot \nabla \right) H_\mathrm{S} \right] \\ &= \frac{2}{3} \delta^{\alpha\beta} \Delta^2 S^2 - C_\mathrm{S} \Delta^2 S \mathcal{S}^{\alpha\beta} + C_{\eta\mathrm{H}} \frac{\Delta^2}{S} \left[ \overline{\omega}^\alpha \frac{\partial H_\mathrm{S}}{\partial x^\beta} + \overline{\omega}^\beta \frac{\partial H_\mathrm{S}}{\partial x^\alpha} - \frac{2}{3} \delta^{\alpha\beta} \left( \overline{\boldsymbol{\omega}} \cdot \nabla \right) H_\mathrm{S} \right]\end{aligned}$$

SGS turbulent helicity ($H_\mathrm{S}$) equation

$$\left( \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \right) H_\mathrm{S} = -\mathcal{R}^{ab} \frac{\partial \overline{\omega}^b}{\partial x^a} + \overline{\omega}^a \frac{\partial \mathcal{R}^{ab}}{\partial x^b} - \epsilon_\mathrm{HS} + \nabla \cdot \left( K \overline{\boldsymbol{\omega}} + \mathbf{T}_\mathrm{HS} \right)$$

$$\epsilon_\mathrm{HS} = C_{\epsilon\mathrm{H}} \frac{H_\mathrm{S}}{1/S} = C_{\epsilon\mathrm{H}} H_\mathrm{S} S \qquad \mathbf{T}_\mathrm{HS} = \frac{\nu_\mathrm{S}}{\sigma_\mathrm{HS}} \nabla H_\mathrm{S}$$

Local equilibrium of the SGS energy $\quad$ Production ~ Dissipation $\quad -\mathcal{R}^{ab} \dfrac{\partial \overline{u}^b}{\partial x^a} \simeq \epsilon_\mathrm{S}$

$$\epsilon_\mathrm{S} \simeq -\mathcal{R}^{ab} \frac{\partial \overline{u}^b}{\partial x^a} \simeq \nu_\mathrm{S} S^2 = \left( C_\mathrm{S} \Delta \right)^2 S S^2 = \left( C_\mathrm{S} \Delta \right)^2 S^3$$

## Zero-equation model (Same level as Smagorinsky model)

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right) \overline{\mathbf{u}} = -\nabla \overline{p} - \nabla \cdot \mathcal{R} + \nu \nabla^2 \overline{\mathbf{u}}$$

$$\mathcal{R}^{\alpha\beta} = \frac{2}{3} K_\mathrm{S} \delta^{\alpha\beta} - \nu_\mathrm{S} S^{\alpha\beta} + \eta_\mathrm{S} \left[\overline{\omega}^\alpha \frac{\partial H_\mathrm{S}}{\partial x^\beta} + \overline{\omega}^\beta \frac{\partial H_\mathrm{S}}{\partial x^\alpha} - \delta^{\alpha\beta} \left(\overline{\boldsymbol{\omega}} \cdot \nabla\right) H_\mathrm{S}\right]$$

$$\nu_\mathrm{S} = \left(C_\mathrm{S} \Delta\right)^2 S \qquad \eta_\mathrm{S} = C_\eta \frac{\Delta^2}{S}$$

$$K_\mathrm{S} = \Delta^2 S^2$$

$$H_\mathrm{S} = \frac{1}{C_{\epsilon \mathrm{H}} S} \left[\nu_\mathrm{S} S^{ab} \frac{\partial \overline{\omega}^b}{\partial x^a} - \overline{\omega}^a \frac{\partial}{\partial x^b} \nu_\mathrm{S} S^{ab} + \frac{2}{3} \left(\overline{\boldsymbol{\omega}} \cdot \nabla\right) K_\mathrm{S}\right]$$

requires test using DNS

Local equilibrium of the SGS helicity $-\mathcal{R}^{ab} \frac{\partial \overline{\omega}^b}{\partial x^a} + \overline{\omega}^a \frac{\partial R^{ab}}{\partial x^b} \simeq \epsilon_\mathrm{HS}$ Production ~ Dissipation

$$\epsilon_\mathrm{HS} = C_{\epsilon \mathrm{H}} H_\mathrm{S} S \qquad \text{with} \qquad S \equiv \sqrt{\frac{1}{2} \left(\mathcal{S}^{ab}\right)^2}$$

$$\mathcal{R}^{ab} = \frac{2}{3} K_\mathrm{S} \delta^{\alpha\beta} - \nu_\mathrm{S} \mathcal{S}^{ab} + \cdots$$

$$H_\mathrm{S} = \frac{1}{C_{\epsilon \mathrm{H}} S} \epsilon_\mathrm{HS} \simeq \frac{1}{C_{\epsilon \mathrm{H}} S} \left[\nu_\mathrm{S} \mathcal{S}^{ab} \frac{\partial \overline{\omega}^b}{\partial x^a} - \overline{\omega}^a \frac{\partial}{\partial x^b} \nu_\mathrm{S} \mathcal{S}^{ab} + \frac{2}{3} \left(\overline{\boldsymbol{\omega}} \cdot \nabla\right) K_\mathrm{S}\right]$$

# Magnetohydrodynamic Case

Grid-Scale (GS) velocity equation

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\right)\overline{\mathbf{u}} = -\nabla\overline{p} - \nabla \cdot \boldsymbol{\mathcal{R}} + \overline{\mathbf{j}} \times \overline{\mathbf{b}} + \nu\nabla^2\overline{\mathbf{u}}$$

SGS stress

$$\begin{aligned}\mathcal{R}^{\alpha\beta} &\equiv \overline{u^\alpha u^\beta} - \overline{u}^\alpha\overline{u}^\beta - \left(\overline{b^\alpha b^\beta} - \overline{b}^\alpha\overline{b}^\beta\right) \\ &= \frac{2}{3}K_{\mathrm{S}}\delta^{\alpha\beta} - \nu_{\mathrm{S}}\mathcal{S}^{\alpha\beta} + \cancel{\nu_{\mathrm{M}}\mathcal{M}^{\alpha\beta}}\end{aligned}$$

GS strain rate

$$\mathcal{S}^{\alpha\beta} \equiv \frac{\partial\overline{u}^\alpha}{\partial x^\beta} + \frac{\partial\overline{u}^\beta}{\partial x^\alpha} \qquad \mathcal{M}^{\alpha\beta} \equiv \frac{\partial\overline{b}^\alpha}{\partial x^\beta} + \frac{\partial\overline{b}^\beta}{\partial x^\alpha}$$

Grid-Scale (GS) magnetic-field equation

$$\frac{\partial\overline{\mathbf{b}}}{\partial t} = \nabla \times \left(\overline{\mathbf{u}} \times \overline{\mathbf{b}}\right) + \nabla \times \mathbf{E}_{\mathrm{M}} + \eta\nabla^2\overline{\mathbf{b}}$$

SGS electromotive force

$$\begin{aligned}\mathbf{E}_{\mathrm{M}} &\equiv \overline{\mathbf{u} \times \mathbf{b}} - \overline{\mathbf{u}} \times \overline{\mathbf{b}} \\ &= -\beta\nabla \times \overline{\mathbf{b}} + \cancel{\alpha\overline{\mathbf{b}} + \gamma\nabla \times \overline{\mathbf{u}}}\end{aligned}$$

# MHD Counterpart of the Smagorinsky Model

## Local equilibrium of the SGS energy

Production ~ Dissipation $P_\mathrm{S} \simeq \varepsilon_\mathrm{S}$

Production
$$
\begin{aligned}
P_\mathrm{S} &= -\mathcal{R}^{ab}\frac{\partial \overline{u}^b}{\partial x^a} - \mathbf{E}_\mathrm{M}\cdot\bar{\mathbf{j}} \\
&\simeq \nu_\mathrm{S}(\mathcal{S}^{ab})^2/2 + \beta_\mathrm{S}\bar{\mathbf{j}}^2 \\
&\simeq \nu_\mathrm{S}\left[(\mathcal{S}^{ab})^2/2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right]
\end{aligned}
$$

Turbulent magnetic Prandtl number

$$\sigma_\mathrm{m} = \nu_\mathrm{S}/\beta_\mathrm{S}$$

Dissipation
$$\varepsilon_\mathrm{S} \simeq C_{\varepsilon\mathrm{S}}\frac{K_\mathrm{S}^{3/2}}{\Delta}$$

$$\nu_\mathrm{S} = C_{\nu\mathrm{S}}K_\mathrm{S}^{1/2}\Delta$$

$$K_\mathrm{S}^{3/2} = \left(\frac{\nu_\mathrm{S}}{C_{\nu\mathrm{S}}\Delta}\right)^3$$

$$-\mathcal{R}^{ab}\frac{\partial \overline{u}^b}{\partial x^a} - \mathbf{E}_\mathrm{M}\cdot\bar{\mathbf{j}}$$

$$\simeq \nu_\mathrm{S}(\mathcal{S}^{ab})^2/2 + \beta_\mathrm{S}\bar{\mathbf{j}}^2 = \nu_\mathrm{S}\left[(\mathcal{S}^{ab})^2/2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right]$$

$$\varepsilon_\mathrm{S} \simeq C_{\varepsilon\mathrm{S}}\frac{K_\mathrm{S}^{3/2}}{\Delta} = C_{\varepsilon\mathrm{S}}\frac{1}{\Delta}\left(\frac{\nu_\mathrm{S}}{C_{\nu\mathrm{S}}\Delta}\right)^3 = \boxed{\frac{C_{\varepsilon\mathrm{S}}}{C_{\nu\mathrm{S}}^3}\frac{\nu_\mathrm{S}^3}{\Delta^4}} \qquad \varepsilon_\mathrm{S} \simeq \frac{\nu_\mathrm{S}^3}{(C_\mathrm{S}\Delta)^4} \qquad C_\mathrm{S} \equiv \left(\frac{C_{\nu\mathrm{S}}^3}{C_{\varepsilon\mathrm{S}}}\right)^{1/4}$$

Local equilibrium $P_\mathrm{S} \simeq \varepsilon_\mathrm{S}$

$$\nu_\mathrm{S}\left[(\mathcal{S}^{ab})^2/2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right] \simeq \frac{C_{\varepsilon\mathrm{S}}}{C_{\nu\mathrm{S}}^3}\frac{\nu_\mathrm{S}^3}{\Delta^4}$$

$$\nu_\mathrm{S}^2 \simeq \frac{C_{\nu\mathrm{S}}^3}{C_{\varepsilon\mathrm{S}}}\Delta^4\left[(\mathcal{S}^{ab})^2/2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right]$$

$\longrightarrow$ Smagorinsky model for MHD

$$\boxed{\nu_\mathrm{S} = (C_\mathrm{S}\Delta)^2\left[\frac{1}{2}\left(\mathcal{S}^{ab}\right)^2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right]^{1/2}}$$

$$\boxed{\varepsilon_\mathrm{S} \simeq \frac{\nu_\mathrm{S}^3}{(C_\mathrm{S}\Delta)^4} = (C_\mathrm{S}\Delta)^2\left[\frac{1}{2}\left(\mathcal{S}^{ab}\right)^2 + \frac{1}{\sigma_\mathrm{m}}\bar{\mathbf{j}}^2\right]^{3/2}}$$

# Local equilibrium of the SGS cross helicity

Production ~ Dissipation $\quad P_{W\mathrm{S}} \simeq \varepsilon_{W\mathrm{S}}$

$$\begin{cases} P_{W\mathrm{S}} = -\mathcal{R}_{ab} \dfrac{\partial \overline{b}^a}{\partial x^b} - \mathbf{E}_{\mathrm{M}} \cdot \overline{\boldsymbol{\omega}} \\[2ex] \varepsilon_{W\mathrm{S}} = C_{W\mathrm{S}} \dfrac{W_{\mathrm{S}}}{\tau_{\mathrm{S}}} = C_{W\mathrm{S}} \dfrac{K_{\mathrm{S}}^{1/2}}{\Delta} W_{\mathrm{S}} \end{cases} \qquad \left[ \tau_{\mathrm{S}} = \frac{\Delta}{K_{\mathrm{S}}^{1/2}} \right]$$

Production
$$\begin{aligned} P_W &= -\mathcal{R}_{ab} \frac{\partial \overline{b}^a}{\partial x^b} - \mathbf{E}_{\mathrm{M}} \cdot \overline{\boldsymbol{\omega}} \\ &= +\nu_{\mathrm{S}} \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \beta_{\mathrm{S}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \\ &\simeq \nu_{\mathrm{S}} \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right) \end{aligned}$$

Dissipation
$$\varepsilon_{W\mathrm{S}} = C_{W\mathrm{S}} \frac{K_{\mathrm{S}}^{1/2}}{\Delta} W_{\mathrm{S}} = \frac{C_{W\mathrm{S}}}{C_{\nu\mathrm{S}}} \frac{\nu_{\mathrm{S}}}{\Delta^2} W_{\mathrm{S}} \qquad \left[ K_{\mathrm{S}}^{1/2} = \frac{\nu_{\mathrm{S}}}{C_{\nu\mathrm{S}} \Delta} \right]$$

$$P_{W\mathrm{S}} \simeq \nu_{\mathrm{S}} \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right)$$

$$\varepsilon_{W\mathrm{S}} \simeq \frac{C_{W\mathrm{S}}}{C_{\nu\mathrm{S}}} \frac{\nu_{\mathrm{S}}}{\Delta^2} W_{\mathrm{S}}$$

Turbulent magnetic Prandtl number

$$\sigma_{\mathrm{m}} = \nu_{\mathrm{S}} / \beta_{\mathrm{S}}$$

Local equilibrium $\quad P_{W\mathrm{S}} \simeq \varepsilon_{W\mathrm{S}}$

$$\nu_{\mathrm{S}} \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right) \simeq \frac{C_{W\mathrm{S}}}{C_{\nu\mathrm{S}}} \frac{\nu_{\mathrm{S}}}{\Delta^2} W_{\mathrm{S}}$$

$$\boxed{W_{\mathrm{S}} \simeq \frac{C_{\nu\mathrm{S}}}{C_{W\mathrm{S}}} \Delta^2 \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right)}$$

$$\begin{aligned} \longrightarrow \quad \varepsilon_{W\mathrm{S}} &\simeq \frac{C_{W\mathrm{S}}}{C_{\nu\mathrm{S}}} \frac{\nu_{\mathrm{S}}}{\Delta^2} W_{\mathrm{S}} \\ &= \nu_{\mathrm{S}} \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right) \\ &= \boxed{(C_{\mathrm{S}} \Delta)^2 \left[ \frac{1}{2} \left( \mathcal{S}^{ab} \right)^2 + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}}^2 \right]^{1/2} \left( \frac{1}{2} \mathcal{S}^{ab} \mathcal{M}^{ab} + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}} \right)} \end{aligned}$$

Smagorinsky model for MHD

$$\nu_{\mathrm{S}} = (C_{\mathrm{S}} \Delta)^2 \left[ \frac{1}{2} \left( \mathcal{S}^{ab} \right)^2 + \frac{1}{\sigma_{\mathrm{m}}} \bar{\mathbf{j}}^2 \right]^{1/2}$$

$$C_{\mathrm{S}} \equiv \left( \frac{C_{\nu\mathrm{S}}^3}{C_{\epsilon\mathrm{S}}} \right)^{1/4}$$

# Helicity-related Terms in MHD Case

Turbulent MHD residual helicity

$$H_{\mathrm{R}} \equiv \overline{-\mathbf{u}' \cdot \boldsymbol{\omega}' + \mathbf{b}' \cdot \mathbf{j}'}$$

$H_{\mathrm{R}}$ equation

$$\begin{aligned}\frac{DH_{\mathrm{R}}}{Dt} &= C_{\mathrm{HR}} \frac{K}{\varepsilon} K_{\mathrm{R}} \left( \mathcal{M}^{ab} \frac{\partial J^b}{\partial x^a} - \mathcal{S}^{ab} \frac{\partial \Omega^b}{\partial x^a} \right) \\ &\quad -\frac{1}{2} \frac{\partial \mathcal{R}^{ab}}{\partial x^a} \Omega^b - C_{\mathrm{HB}} \frac{\varepsilon^2}{K^3} \mathbf{E}_{\mathrm{M}} \cdot \mathbf{B} - C_{\mathrm{H}} \frac{\varepsilon}{K} H_{\mathrm{R}} \\ &\quad +\nabla \cdot \left[ -\frac{1}{2} \left( K + K_{\mathrm{R}} \right) \boldsymbol{\Omega} + \frac{\nu_{\mathrm{K}}}{\sigma_{\mathrm{H}}} \nabla H_{\mathrm{R}} \right]\end{aligned}$$

## Local equilibrium

$$P_{H_\mathrm{R}} \simeq \varepsilon_{H_\mathrm{R}}$$

$$C_\mathrm{HR}\frac{K}{\varepsilon}K_\mathrm{R}\left(\mathcal{M}^{ab}\frac{\partial J^b}{\partial x^a} - \mathcal{S}^{ab}\frac{\partial \Omega^b}{\partial x^a}\right)$$
$$-\frac{1}{2}\frac{\partial \mathcal{R}^{ab}}{\partial x^a}\Omega^b - C_\mathrm{HB}\frac{\varepsilon^2}{K^3}\mathbf{E}_\mathrm{M}\cdot\mathbf{B} + \nabla\cdot\left[-\frac{1}{2}\left(K+K_\mathrm{R}\right)\boldsymbol{\Omega}\right]$$
$$\simeq C_\mathrm{H}\frac{\varepsilon}{K}H_\mathrm{R}$$

## Equipartition

$$K_\mathrm{R} = 0$$

$$C_\mathrm{H}\frac{K}{\varepsilon}H_\mathrm{R} \simeq -\frac{1}{2}\frac{\partial \mathcal{R}^{ab}}{\partial x^a}\Omega^b - C_\mathrm{HB}\frac{\varepsilon^2}{K^3}\mathbf{E}_\mathrm{M}\cdot\mathbf{B} - \frac{1}{2}\boldsymbol{\Omega}\cdot\nabla K$$

$$H_\mathrm{R} \simeq -\frac{1}{C_\mathrm{H}}\frac{K}{\varepsilon}\left(\frac{1}{2}\frac{\partial \mathcal{R}^{ab}}{\partial x^a}\Omega^b + C_\mathrm{HB}\frac{\varepsilon^2}{K^3}\mathbf{E}_\mathrm{M}\cdot\mathbf{B} + \frac{1}{2}\boldsymbol{\Omega}\cdot\nabla K\right)$$

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial\tau}+\nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$

$$-2iM_{\alpha ab}(\mathbf{k})\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}u'_{0a}(\mathbf{p};\tau)u'_{S1b}(\mathbf{q};\tau)$$

$$=\quad -D_{\alpha b}(\mathbf{k})u'_{0a}(\mathbf{k};\tau)\frac{\partial U_b}{\partial X_a}-D_{\alpha a}(\mathbf{k})\frac{Du'_{0a}(\mathbf{k};\tau)}{DT_{\mathrm{I}}}$$

$$+2M_{\alpha ab}(\mathbf{k})\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\frac{q_b}{q^2}u'_{0a}(\mathbf{p};\tau)\frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$

$$-D_{\alpha d}(\mathbf{k})M_{abcd}(\mathbf{k})\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\frac{\partial}{\partial X_{\mathrm{I}c}}\left(u'_{0a}(\mathbf{p};\tau)u'_{0b}(\mathbf{q};\tau)\right)$$

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial\tau}+\nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$

$$-2iM^{\alpha ab}(\mathbf{k})\iint\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}u'_{0a}(\mathbf{p};\tau)G'_{b\beta}(\mathbf{q};\tau,\tau')=D_{\alpha\beta}(\mathbf{k})\delta(\tau-\tau')$$

$$R_{\alpha\beta}(\mathbf{k})=\frac{\left\langle u'_{\alpha}(\mathbf{k};\tau)u'_{\beta}(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}$$

$$=\frac{\left\langle u'_{0\alpha}(\mathbf{k};\tau)u'_{0\beta}(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}+\delta\left(\frac{\left\langle u'_{1\alpha}(\mathbf{k};\tau)u'_{0\beta}(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}+\frac{\left\langle u'_{0\alpha}(\mathbf{k};\tau)u'_{1\beta}(\mathbf{k}';\tau)\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}\right)+O(\delta^2)$$

$$\frac{\left\langle u'_{0\alpha}(\mathbf{k};\tau)\,u'_{0\beta}(\mathbf{k}';\tau')\right\rangle}{\delta(\mathbf{k}+\mathbf{k}')}=D_{\alpha\beta}(\mathbf{k})\,Q(k;\tau,\tau')+\frac{i}{2}\frac{k_a}{k^2}\varepsilon_{\alpha\beta a}H(k;\tau,\tau')$$